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FIXED POINTS OF GENERALIZED GóRNICKI MAPS

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ABSTRACT

In this paper, we define generalized Geraghty - Górnicki map, generalized Boyd and Wong - Górnicki map, and generalized weakly Górnicki map and prove the existence and uniqueness of fixed points of these maps in complete metric spaces. These maps are not necessarily be continuous. Examples are provided in support of our results. Our results generalize some of the existing results.

Keywords: Asymptotically regular, complete metric space, k-continous, orbitally continuous, fixed point.

I. Introduction

Fixed point theory has been an attractive field of research to many researchers since 1922 with the famous Banach contraction principle [3], a technique that provides to solve variety of principle problems in mathematical sciences and engineering. Subsequently, this result was extended and generalized by several authors using various contraction/contractive conditions.

A mapping T on a metric space (X,d) is called Kannan map if there exists $\alpha \in \left[0,\frac{1}{2}\right)$ such that

$$d(Tx,Ty) \leq \alpha\big(d(x,Tx)+d(y,Ty)\big)$$

for all $x,y \in X$. In the year 1968, Kannan [15] proved that if (X,d) is complete and T is a Kannan mapping on X, then T has a unique fixed point in X. Kannan [16] provided examples which show that Kannan's fixed point theorem is independent of the Banach contraction principle and Kannan mapping need not be continuous. Based Kannan's fixed point some generalizations are made by Górnicki's [14]. Several authors generalized Kannan's fixed point theorem, see [1], [6], [7], [9]-[13], [15], and [17]-[24].

In 2019, Górnicki [13] considered the following type of mappings in metric spaces, which we call Górnicki maps.

Theorem 1.1. Let (X,d) be a metric space and $T: X \to X$ is a continuous asymptotically regular mapping and if there exists $0 \leq M < 1$ and $0 \leq K < +\infty$ such that

$$d(Tx,Ty) \leq M\, d(x,y) + K\,\{d(x,Tx)+d(y,Ty)\} \qquad (1.1)$$

for all $x,y \in X$, then T has a unique fixed point $p \in X$ and $T^n x \to p$ for each $x \in X$.

A map T that satisfies condition (1.1) is called a Górnicki map. Here we note that such T need not be continuous. For more details, we refer [14].

Definition 1.2. [7] A self map T of a metric space (X,d) is said to be asymptotically regular if $\lim_{n\to\infty} d(T^{n+1}x, T^n x) = 0$ for all $x \in X$.

Definition 1.3. (Ćirić [8]) Let T be a self map of a metric space (X,d). Let $x \in X$ then the set $O(T,x) = \{T^n x : n = 0,1,2,\ldots\}$ is called the orbit of T at x. T is called orbitally continuous at a point $z \in X$ if for any sequence $\{x_n\} \subset O(T,x)$ for some $x \in X$, $x_n \to z$ implies $Tx_n \to Tz$ as $n \to \infty$.

Here we note that every continuous self map of X is orbitally continuous but its converse is not true [8].

The weaker form of continuity is defined by Pant and Pant [20] as follows: a self mapping T of a metric space (X,d) is called k-continuous, $k = 1,2,\ldots$, if $T^k x_n \to Tz$ whenever $\{x_n\}$ is a sequence in X such that $T^{k-1}x_n \to z$ as $n \to \infty$. For more details, we refer Pant and Pant [20] and Górnicki [14].

In 2019, Bisht [4] and Górnicki [14] proved the following theorem.

Theorem 1.4. If (X, d) is a complete metric space and $T : X \to X$ is an asymptotically regular mapping and if there exists $0 \le M < 1$ and $0 \le K < +\infty$ such that satisfying condition (1.1), then T has a unique fixed point $p \in X$ provided T is either k-continuous for some $k \ge 1$ or orbitally continuous. Additionally, $T^n x \to p$ for each $x \in X$.

Notation 1.5. Let $\mathcal{S} = \{\alpha: [0, \infty) \to [0, 1)/\alpha(\tau_n) \to 1 \Rightarrow \tau_n \to 0\}$. (We do not assume that α is continuous in any sense).

Theorem 1.6. [14] Let (X, d) be a metric space. We considered a new type of mappings $T : X \to X$ to satisfy the following condition: there exists $\alpha \in \mathcal{S}$, $0 \le K < \infty$ such that

$$d(Tx, Ty) \le \alpha\big(d(x, y)\big). d(x, y) + K.\{d(x, Tx) + d(y, Ty)\} \quad (1.6.1)$$

for all $x, y \in X$. If T is k-continuous for some $k \ge 1$ or T is orbitally continuous, then T has a unique fixed point $z \in X$ and for each $x \in X, T^n x \to z \, as \, n \to \infty$.

We call a map that satisfies (1.6.1) is called a Geraghty - Górnicki map.

The next result is inspired by theorem of Boyd and Wong [5].

A mapping T satisfying $d(T x, T y) \le \varphi(d(x, y)), \varphi(\tau) < \tau$ for each $\tau > 0$ may not posses a fixed point unless some additional condition is assumed on φ. Boyd and Wong [5] assumed φ to be upper semi-continuous from the right.

Theorem 1.7. [14] Let (X, d) be a metric space. Assume that $T : X \to X$ satisfies the following condition: there exists $\alpha \in \mathcal{S}$, $0 \le k < \infty$ such that

$$d(Tx, Ty) \le \varphi\big(d(x, y)\big) + K.\{d(x, Tx) + d(y, Ty)\} \quad (1.7.1)$$

for all $x, y \in X$. If T is k-continuous for some $k \ge 1$ or T is orbitally continuous, then T has a unique fixed point $z \in X$ and for each $x \in X, T^n x \to z \, as \, n \to \infty$.

We call a map that satisfies (1.7.1) is called a Boyd and Wong - Górnicki map.

To prove our main results, we use the following lemma.

Lemma 1.8. [2] Suppose (X, d) be a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m_i\}$ and $\{n_i\}$ with $m_i > n_i > i$ such that $d\big(x_{m_i}, x_{n_i}\big) \ge \epsilon$, $d\big(x_{m_i-1}, x_{n_i}\big) < \epsilon$ and

(i) $\lim_{i\to\infty} d\big(x_{m_i-1}, x_{n_i+1}\big) = \epsilon;$

(ii) $\lim_{i\to\infty} d\big(x_{m_i}, x_{n_i}\big) = \epsilon;$

$(iii) d\big(x_{m_i-1}, x_{n_i}\big) = \epsilon;$

II. Fixed points of weakly Górnicki map

In the following, we define generalized weakly Górnicki map.

Definition 2.1. Let (X, d) be a metric space and $T: X \to X$. If there exists $\varphi \in \Phi$, and $\mathrm{K} \ge 0$ such that

$$d(Tx, Ty) \le d(x, y) - \varphi\big(d(x, y)\big) + \mathrm{K}.\{d(x, Tx) + d(y, Ty)\}, \quad (2.1.1.)$$

for each $x, y \in X$, where $\mathbb{R}^+ = [0, \infty)$ and we denote

$\Phi = \{\varphi: \mathbb{R}^+ \to \mathbb{R}^+/\varphi$ is continuous and $\varphi(\tau) = 0$ if and only if $\tau = 0\}$. Then we say that T is a weakly Górnicki map.

Theorem 2.2. Let (X, d) be a complete metric space and $T: X \to X$ an asymptotically regular mapping. Assume that T is a weakly Górnicki map. If T is k-continuous for some $k \ge 1$ or T is orbitally continuous, then T has a unique fixed point.

Proof: Let $x_0 \in X$. We define the sequence $x_{n+1} = Tx_n$, for $n = 0,1,2, \ldots$

If $T^p x = T^{p+1} x$ for some $p \in \mathbb{N}$, then $T(T^p x) = T^p x$ is the fixed point of T.

We now assume, without loss of generality, that $x_{n+1} \ne x_n$ for every $n \in \mathbb{N}$.

By using Lemma 1.8, there exists $\epsilon > 0$ and sequences of positive integers $\{m_i\}$ and $\{n_i\}$ of positive integers with $m_i > n_i > i$ such that $d(x_{m_i}, x_{n_i}) \geq \epsilon$, $d(x_{m_i-1}, x_{n_i}) < \epsilon$ and $(i), (ii)$ and (iii) of Lemma 1.8., hold. We now consider

$$\begin{aligned} d(x_{n_i}, x_{m_i}) &\leq d(x_{n_i}, x_{n_i+1}) + d(x_{n_i+1}, x_{m_i+1}) + d(x_{m_i+1}, x_{m_i}) \\ &\leq d(x_{n_i}, x_{n_i+1}) + d(x_{n_i}, x_{m_i}) - \varphi\left(d(x_{n_i}, x_{m_i})\right) \\ &\qquad + K.\{d(x_{n_i}, x_{n_i+1}) + d(x_{m_i}, x_{m_i+1})\} + d(x_{m_i}, x_{m_i+1}) \\ &= d(x_{n_i}, x_{m_i}) - \varphi\left(d(x_{n_i}, x_{m_i})\right) + (K+1).\left[d(x_{n_i}, x_{n_i+1}) + d(x_{m_i}, x_{m_i+1})\right] \end{aligned}$$

On letting $i \to \infty$ and by using asymptotic regularity and continuity of φ, we obtain $\varphi(\epsilon) = 0$ which imples that $\epsilon = 0$, a contradiction.
Therefore $\{x_n\}$ is a Cauchy sequence.
Since X is complete $\lim_{i\to\infty} x_n = z \in X$.
Suppose T is k –continuous. Let $\{x_n\}$ be a sequence in X such that $T^{k-1}x_n \to z$ as $n \to \infty$.
Then $T^k x_n \to Tz$ as $n \to \infty$.
Also, $T^k x_n = x_{n+k} \to z$ as $n \to \infty$, and hence it follows that $Tz = z$.
Suppose T is orbitally continuous.
Let $\{x_n\}$ be a sequence in $O(T, x)$ such that $x_n \to z$, $z \in X$. Then $x_{n+1} = Tx_n \to Tz$ as $n \to \infty$.
Therefore $Tz = z$.
Therefore z is a fixed point of T.
We now show the uniqueness of z.
Suppose y is another fixed point of T; $Ty = y, Tz = z$.
Suppose $y \neq z$. We consider

$$\begin{aligned} d(z, y) = d(Tz, Ty) &\leq d(z, y) - \varphi(d(z, y)) + K\{d(z, Tz) + d(y, Ty)\} \\ &= d(z, y) - \varphi(d(z, y)) + K\{d(z, z) + d(y, y)\} < d(z, y), \end{aligned}$$

a contradiction.
Therefore z is the unique fixed point of T.
Hence, the theorem follows.

Remark 2.3. Theorem 1.1. follows as a Corollary to Theorem 2.2. by choosing $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = (1 - M)t$, $t \geq 0$ in the inequality 2.1.1.

Example 2.4. Let $X = [0,2]$ with usual metric. Define $T: X \to X$ by $Tx = \begin{cases} 1, & if \quad 0 \leq x \leq 1 \\ 0, & if \quad x > 1 \end{cases}$.

Let $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = \begin{cases} \frac{t}{t+1}, & if\ 0 \leq t \leq 1 \\ \frac{1}{t+1}, & if\ \ t \geq 1 \end{cases}$. Then $\varphi \in \Phi$.

Clearly, T is asymptotically regular.
We now verify the inequality (2.1.1.). Suppose $x \in [0,1]$ and $y > 1$.
If $y - x \in [0, 1]$ implies $\varphi(d(x, y)) = \frac{y-x}{y-x+1}$, then

$$\begin{aligned} d(Tx, Ty) = 1 &\leq 1 - x + y \\ &\leq \frac{y-x}{y-x+1} + 2(1 - x + y) \\ &\leq y - x - \frac{y-x}{y-x+1} + 2(1 - x + y) \\ &= d(x, y) - \varphi(d(x, y)) + K.\{d(x, Tx) + d(y, Ty)\} \text{ with } K = 2. \end{aligned}$$

If $y - x \geq 1$ implies $\varphi(d(x, y)) = \frac{1}{y-x+1}$, then

$$\begin{aligned} d(Tx, Ty) = 1 &\leq 1 - x + y \\ &\leq \frac{1}{y-x+1} + 2(1 - x + y) \end{aligned}$$

$$\leq y - x - \frac{1}{y-x+1} + 2(1 - x + y)$$
$$= d(x,y) - \varphi\big(d(x,y)\big) + K.\{d(x,Tx) + d(y,Ty)\} \text{ with } K = 2$$

So the inequality $(2.1.1.)$ holds with $K = 2$ in this case.
Similarly, the inequality $(2.1.1.)$ holds for the case $y \in [0,1]$ and $x > 1$.
In the other cases, the inequality $(2.1.1.)$ holds trivially.
Then T satisfies the inequality $(2.1.1.)$ and all the hypotheses of Theorem 2.2. and clearly $'1'$ is the unique fixed point of T.

III. Fixed Points of generalized Geraghty- Górnicki map

Here we define generalized Geraghty- Górnicki map.

Definition 3.1. Let (X,d) be a metric space and $T: X \to X$. If there exists $\alpha \in \mathcal{S}$, and $K \geq 0$ such that

$$d(Tx,Ty) \leq \alpha\big(M(x,y)\big).M(x,y) + K.\{d(x,Tx) + d(y,Ty)\}, \qquad (3.1.1.)$$

for each $x,y \in X$ where $M(x,y) = \left\{d(x,y), d(x,Ty), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$. Then we say that T is generalized Geraghty- Górnicki map.

Theorem 3.2. Let (X,d) be a complete metric space and $T: X \to X$ be an asymptotically regular mapping. Assume that T is a generalized Geraghty- Górnicki map. If T is k-continuous for some $k \geq 1$ or T is orbitally continuous, then T has a unique fixed point.

Proof: Let $x_0 \in X$. We define the sequence $x_{n+1} = Tx_n$, for $n = 0,1,2,\ldots$
If $x_{n+1} = x_n$ for some n, then we have $Tx_n = x_n$.
By choosing $z = x_n$, we have $Tz = z$ and the conclusion of the theorem follows.
We now assume, without loss of generality, that $x_{n+1} \neq x_n$ for every $n \in \mathbb{N}$.
By using Lemma 1.8, there exists $\epsilon > 0$ and sequences of positive integers $\{m_i\}$ and $\{n_i\}$ of positive integers with $m_i > n_i > i$ such that $d\big(x_{m_i}, x_{n_i}\big) \geq \epsilon$, $d\big(x_{m_i-1}, x_{n_i}\big) < \epsilon$ and $(i),(ii)$ and (iii) of Lemma 1.8., hold. Now we consider

$$\begin{aligned}
d\big(x_{n_i}, x_{m_i}\big) &\leq d\big(x_{n_i}, x_{n_{i+1}}\big) + d\big(x_{n_{i+1}}, x_{m_{i+1}}\big) + d\big(x_{m_{i+1}}, x_{m_i}\big)\\
&\leq d\big(x_{n_i}, x_{n_{i+1}}\big) + \alpha\Big(M\big(x_{n_i}, x_{m_i}\big)\Big).M\big(x_{n_i}, x_{m_i}\big)\\
&\qquad + K.\{d\big(x_{n_i}, x_{n_{i+1}}\big) + d\big(x_{m_i}, x_{m_{i+1}}\big)\} + d\big(x_{m_i}, x_{m_{i+1}}\big)\\
&= \alpha\left(max\left\{d\big(x_{n_i}, x_{m_i}\big), d\big(x_{n_i}, x_{n_{i+1}}\big), d\big(x_{m_i}, x_{m_{i+1}}\big), \frac{d\big(x_{n_i}, x_{m_{i+1}}\big) + d\big(x_{m_i}, x_{n_{i+1}}\big)}{2}\right\}\right)\\
&\qquad max\left\{d\big(x_{n_i}, x_{m_i}\big), d\big(x_{n_i}, x_{n_{i+1}}\big), d\big(x_{m_i}, x_{m_{i+1}}\big), \frac{d\big(x_{n_i}, x_{m_{i+1}}\big) + d\big(x_{m_i}, x_{n_{i+1}}\big)}{2}\right\}\\
&\qquad + (K+1).\big[d\big(x_{n_i}, x_{n_{i+1}}\big) + d\big(x_{m_i}, x_{m_{i+1}}\big)\big]
\end{aligned}$$

on letting $i \to \infty$ and by using asymptotic regularity we obtain

$$\begin{aligned}
\epsilon &= \lim_{i\to\infty} d\big(x_{n_i}, x_{m_i}\big)\\
&\leq \lim_{i\to\infty} \alpha\left(max\left\{d\big(x_{n_i}, x_{m_i}\big), d\big(x_{n_i}, x_{n_{i+1}}\big), d\big(x_{m_i}, x_{m_{i+1}}\big), \frac{d\big(x_{n_i}, x_{m_{i+1}}\big) + d\big(x_{m_i}, x_{n_{i+1}}\big)}{2}\right\}\right).\epsilon \leq \epsilon
\end{aligned}$$

it follows that

$$\lim_{i\to\infty} \alpha\left(max\left\{d\big(x_{n_i}, x_{m_i}\big), d\big(x_{n_i}, x_{n_{i+1}}\big), d\big(x_{m_i}, x_{m_{i+1}}\big), \frac{d\big(x_{n_i}, x_{m_{i+1}}\big) + d\big(x_{m_i}, x_{n_{i+1}}\big)}{2}\right\}\right) = 1.$$

Thus, by the property of $\alpha \in \mathcal{S}$, we have

$$\lim_{i\to\infty} \alpha\left(max\left\{d\big(x_{n_i}, x_{m_i}\big), d\big(x_{n_i}, x_{n_i+1}\big), d\big(x_{m_i}, x_{m_i+1}\big), \frac{d\big(x_{n_i}, x_{m_i+1}\big) + d\big(x_{m_i}, x_{n_i+1}\big)}{2}\right\}\right) = 0.$$

Thus, we have

$$\lim_{i\to\infty} d\big(x_{n_i}, x_{m_i}\big) = 0,$$

which is a contradiction.
Therefore $\{x_n\}$ is a Cauchy sequence.
Since X is a fixed point of T follow as in the proof of Theorem 2.2.
We now show the uniqueness of z.
Suppose y is another fixed point of T; $Ty = y, Tz = z$.
Suppose $y \neq z$. We consider

$$d(z,y) = d(Tz,Ty) \leq \alpha\big(M(z,y)\big).M(z,y) + K\{d(z,Tz) + d(y,Ty)\}$$
$$= \alpha\left(max\left\{d(z,y), d(z,Tz), d(y,Ty), \frac{d(z,Ty)+d(y,Tz)}{2}\right\}\right).$$
$$max\left\{d(z,y), d(z,Tz), d(y,Ty), \frac{d(z,Ty)+d(y,Tz)}{2}\right\} + K\{d(z,z) + d(y,y)\}$$
$$= \alpha\big(d(z,y)\big).d(z,y) < d(z,y),$$

a contradiction.
Therefore $z = y$ and z is the unique fixed point of T.
Hence the theorem follows.

Example 3.3. Let $X = [0,1]$ with usual metric and $T: X \to X$ by $Tx = \frac{x^2}{2}$, for all $x \in X$. Define $\alpha: [0,\infty) \to [0,\infty)$ by $\alpha(t) = \frac{1}{1+t}, t \geq 0$. Then $\alpha \in \mathcal{S}$. We now verify the inequality (3.1.1).

$Tx = \frac{x^2}{2}$. Then

$$T^2x = T\left(\frac{x^2}{2}\right) = \frac{(x^2)^2}{2^{2+1}}$$
$$T^3x = T\left(\frac{x^4}{2^3}\right) = \frac{(x^2)^4}{2^{2(3)+1}}$$

on continuing this process,

$$T^n x = \frac{x^{2^n}}{2^{2^n-1}};\ T^{n+1}x = \frac{x^{2^{n+1}}}{2^{2^{n+1}-1}};$$

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = \lim_{n\to\infty} |T^n x - T^{n+1}x| = \lim_{n\to\infty}\left|\frac{x^{2^n}}{2^{2^n-1}} - \frac{x^{2^{n+1}}}{2^{2^{n+1}-1}}\right|$$
$$= 2.\lim_{n\to\infty}\left|\frac{x^{2^n}}{2^{2^n}} - \frac{x^{2^{n+1}}}{2^{2^{n+1}}}\right|$$
$$= 2.\lim_{n\to\infty}\left|\frac{x^{2^n}}{2^{2^n}}\left(1 - \frac{x}{2^2}\right)\right|$$
$$= 2.\left(\left(\frac{x}{2}\right)^{2^n}.\left(1 - \frac{x}{2^2}\right)\right) \to 0 \text{ as } n \to \infty$$

Therefore T is asymptotic regular.
Let $x, y \in X$.
Now, $d(x,Tx) + d(y,Ty) = \left|x - \frac{x^2}{2}\right| + \left|y - \frac{y^2}{2}\right|$
Without loss of generality, we assume that $x < y$. Then

$$d(Tx,Ty) = \frac{1}{2}(x^2 - y^2) \leq \frac{1}{2}(x^2 + y - x - y^2) = \frac{x^2}{2} + \frac{y}{2} - \frac{x}{2} - \frac{y^2}{2}$$
$$\leq \left|\frac{x^2}{2} - x\right| + \left|y - \frac{y^2}{2}\right|$$
$$\leq K\left[\left|x - \frac{x^2}{2}\right| + \left|y - \frac{y^2}{2}\right|\right]$$
$$= K\,[d(x,Tx) + d(y,Ty)], \text{ with } K = 1.$$

Therefore the inequality (3.1.1) holds.
Note that T is continuous and satisfies all the hypotheses of Theorem 3.1 and it has a unique fixed point $'0'$.

Example 3.4. Let $X = [0,2]$ with the usual metric. Define $T: X \to X$ by $Tx = \begin{cases} 1, & if\ 0 \le x \le 1 \\ \frac{x}{2}, & if\ 1 < x \le 2 \end{cases}$.

Let $\{x_n\}$ be a sequence in $[0,2]$ such that $Tx_n \to t$ as $n \to \infty$ i.e., $\lim_{n\to\infty} x_n = 2t$.

$\alpha: [0,\infty) \to [0,\infty)$ by $\alpha(t) = \frac{1}{1+t}, t \ge 0$. Then $\alpha \in \mathcal{S}$.

Let $1 < x \le 2$. Then

$Tx = \dfrac{x}{2}$

$T^n x = \frac{x}{2^n}, n = 1,2, \ldots.$

Now $d(T^{n+1}x, T^n x) = \left|\frac{x}{2^{n+1}} - \frac{x}{2^n}\right| = \frac{x}{2^n} - \frac{x}{2^{n+1}} \to 0$ as $n \to \infty$.

Therefore T is asymptotically regular on $(1,2]$.

Clearly T is asymptotically regular on $[0,1]$, hence T is asymptotically regular on X.

Let us now verify the inequality (3.1.1).

Case (I): Let $x \in [0,1]$ and $y \in (1,2]$.

Then $d(x, Ty) = d\left(x, \frac{y}{2}\right) = \left|x - \frac{y}{2}\right| = \begin{cases} \frac{y}{2} - x, & x < \frac{y}{2} \\ x - \frac{y}{2}, & \frac{y}{2} < x \end{cases}$

Subcase (i): $x < \frac{y}{2}$,

In this case,

$$M(x,y) = max\left\{y - x, 1 - x, \frac{y}{2}, \frac{1}{2}\left[\frac{y}{2} - x + y - 1\right]\right\}$$
$$= max\left\{y - x, 1 - x, \frac{y}{2}, \frac{3y-2x-2}{4}\right\}.$$

Since $y > 1$ and $x < \frac{y}{2}$, we have

$$M(x,y) = max\left\{y - x, \frac{3y - 2x - 2}{4}\right\} \qquad (3.4.1)$$

$M(x,y) = y - x$, then

$$d(Tx,Ty) = 1 - \frac{y}{2} \le 1 + \frac{y}{2}$$
$$\le \frac{y-x}{1+y-x} + 1 - x + \frac{y}{2}$$
$$= \alpha\big(M(x,y)\big).M(x,y) + K\,[d(x,Tx) + d(y,Ty)] \text{ with } K = 1.$$

If $\frac{3y-2x-2}{4}$ is the maximum of (3.4.1), i.e., $M(x,y) = \frac{3y-2x-2}{4}$ then

$$d(Tx,Ty) = 1 - \frac{y}{2} \le 1 + \frac{y}{2}$$
$$\le \left(\frac{4}{3y-2x+2}\right).\frac{3y-2x-2}{4} + 1 + \frac{y}{2}$$
$$\le \frac{3y-2x-2}{3y-2x+2} + 1 + \frac{y}{2}$$
$$\le \frac{3y-2x-2}{3y-2x+2} + 1 - x + \frac{y}{2}$$
$$= \alpha\big(M(x,y)\big).M(x,y) + K\,[d(x,Tx) + d(y,Ty)] \text{ with } K = 1.$$

Subcase (ii): $x > \frac{y}{2}$

In this case,

$$M(x,y) = max\left\{y - x, 1 - x, \frac{y}{2}, \frac{1}{2}\left[x - \frac{y}{2} + y - 1\right]\right\}$$
$$= max\left\{y - x, 1 - x, \frac{y}{2}, \frac{2x+y-2}{4}\right\}$$ Since $y \ge 1$ and $x > \frac{y}{2}$, we have

$$M(x,y) = max\left\{\frac{y}{2}, \frac{2x+y-2}{4}\right\} \qquad (3.4.2)$$

If $M(x,y) = \frac{y}{2}$ then we verify (3.2.1) as follows

$$d(Tx,Ty) = 1 - \frac{y}{2} \le 1 + \frac{y}{2}$$
$$\le \frac{y}{2+y} + 1 + \frac{y}{2}$$
$$\le \frac{y}{2+y} + 1 - x + \frac{y}{2}$$
$$= \alpha\big(M(x,y)\big).M(x,y) + K\,[d(x,Tx) + d(y,Ty)] \text{ with } K = 1.$$

Suppose $M(x,y) = \frac{2x+y-2}{4}$. Then

$$d(Tx,Ty) = 1 - \frac{y}{2} \le 1 + \frac{y}{2}$$
$$\le \frac{2x+y-2}{2x+y+2} + 1 + \frac{y}{2}$$
$$\le \frac{2x+y-2}{2x+y+2} + 1 - x + \frac{y}{2}$$
$$= \alpha\big(M(x,y)\big).M(x,y) + K\,[d(x,Tx) + d(y,Ty)] \text{ with } K = 1.$$

Case (II): Let $x, y \in (1,2]$

$$d(Tx,Ty) = d\left(\frac{x}{2},\frac{y}{2}\right) = \left|\frac{x}{2} - \frac{y}{2}\right|$$
$$d(x,y) = |x-y|$$
$$d(x,Tx) = d\left(x,\frac{x}{2}\right) = \left|x - \frac{x}{2}\right| = \frac{x}{2}$$
$$d(y,Ty) = d\left(y,\frac{y}{2}\right) = \left|y - \frac{y}{2}\right| = \frac{y}{2}$$
$$d(x,Ty) = d\left(x,\frac{y}{2}\right) = \left|x - \frac{y}{2}\right|$$
$$d(y,Tx) = d\left(y,\frac{x}{2}\right) = \left|y - \frac{x}{2}\right|$$

Therefore $d(Tx,Ty) = \frac{1}{2}|x-y| \le K[d(x,Tx) + d(y,Ty)]$ with $K = 1$.
So, clearly in this case the inequality (3.1.1) holds.

Case (III): Let $y \in [0,1]$ and $x \in (1,2]$

$$d(Tx,Ty) = \left|1 - \frac{x}{2}\right| = 1 - \frac{x}{2}$$
$$d(x,y) = |x-y| = x - y$$
$$d(x,Tx) = \left|x - \frac{x}{2}\right| = \frac{x}{2}$$
$$d(y,Ty) = d(y,1) = |y-1| = 1 - y$$
$$d(x,Ty) = d(x,1) = |x-1| = x - 1$$
$$d(y,Tx) = d\left(y,\frac{x}{2}\right) = \left|y - \frac{x}{2}\right| = \begin{cases} \frac{x}{2} - y, & y \le \frac{x}{2} \\ y - \frac{x}{2}, & \frac{x}{2} \le y \end{cases}$$

Subcase (i): $\frac{x}{2} < y$

In this case, $M(x,y) = max\left\{x - y, \frac{x}{2}, 1 - y, \frac{x-1+\frac{x}{2}-y}{2}\right\}$

$$= max\left\{x - y, \frac{x}{2}, 1 - y, \frac{3x-2y-2}{4}\right\} \text{ Since } y < x \text{ and } \frac{x}{2} < y$$
$$M(x,y) = \frac{x}{2}$$

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$$d(Tx,Ty) = 1-\frac{x}{2}$$
$$\leq \frac{x}{2+x}+\frac{x}{2}+1-y$$
$$= \alpha\big(M(x,y)\big).M(x,y) + K\,[d(x,Tx)+d(y,Ty)] \text{ with } K=1.$$

Subcase (ii): $\frac{x}{2} > y$

In this case, $M(x,y) = max\left\{x-y, \frac{x}{2}, 1-y, \frac{x-1+y-\frac{x}{2}}{2}\right\}$

$$= max\left\{x-y, \frac{x}{2}, 1-y, \frac{x-2+2y}{4}\right\} \text{ Since } y<x \text{ and } \frac{x}{2}>y$$
$$M(x,y) = \frac{x}{2}$$
$$d(Tx,Ty) = 1-\frac{x}{2}$$
$$\leq \frac{x}{2+x}+\frac{x}{2}+1-y$$
$$= \alpha\big(M(x,y)\big).M(x,y) + K\,[d(x,Tx)+d(y,Ty)] \text{ with } K=1.$$

So, clearly in this case the inequality (3.1.1) holds.
In all cases T satisfies the inequality (3.1.1) with $K = 1$ and hypotheses of Theorem 3.2 and it has unique fixed point $'1'$.

IV. Fixed points of Boyd and Wong - Górnicki map

Here we define generalized Boyd and Wong- Górnicki map.

Definition 4.1. [13] Let $\mathcal{U}$ denote the class of all mappings $\varphi: [0,\infty) \to [0,\infty)$ satisfying

(i) $\varphi(t) < t$ for all $t > 0$

(ii) φ is upper semi-continuous, that is, $t_n \to t \geq 0 \Rightarrow limsup_{n\to\infty}\varphi(t_n) \leq \varphi(t)$.

Definition 4.2. Let (X,d) be a complete metric space. Let $T: X \to X$. If there exists $\in \mathcal{U}$, $0 \leq K < \infty$, such that

$$d(Tx,Ty) \leq \varphi\big(M(x,y)\big) + K.\{d(x,Tx)+d(y,Ty)\} \qquad (4.2.1)$$

for each $x,y \in X$ where $M(x,y) = max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$ then we call T is a generalized Boyd and Wong- Górnicki map.

Theorem 4.3. Let (X,d) be a complete metric space and $T: X \to X$ an asymptotically regular mapping. Assume that T is a generalized Boyd and Wong- Górnicki map. If T is k-continuous for some $k \geq 1$ or T is orbitally continuous, then T has a unique fixed point $z \in X$ and for each $x \in X$, $T^n x \to z \text{ as } n \to \infty$.

Proof: Let $x \in X$ and define $x_n = T^n x,\ n = 1,2,\ldots$
If $T^p x = T^{p+1}x$ for some $p \in \mathbb{N}$, then $T(T^p x) = T^p x$, so $T^p x$ is a fixed point of T.
Suppose $T^{n+1}x \neq T^n x$, for all $n \geq 0$.
By using Lemma 1.8, there exists $\epsilon > 0$ and sequences of positive integers $\{m_i\}$ and $\{n_i\}$ of positive integers with $m_i > n_i > i$ such that $d(x_{m_i}, x_{n_i}) \geq \epsilon$, $d(x_{m_i-1}, x_{n_i}) < \epsilon$ and $(i),(ii)$ and (iii) of Lemma 1.8., hold. Now we consider

$$d(x_{n_i},x_{m_i}) \leq d(x_{n_i},x_{n_i+1}) + d(x_{n_i+1},x_{m_i+1}) + d(x_{m_i+1},x_{m_i})$$
$$\leq d(x_{n_i},x_{n_i+1}) + \varphi\left(M(x_{n_i},x_{m_i})\right) + K.\{d(x_{n_i},x_{n_i+1}) + d(x_{m_i},x_{m_i+1})\} + d(x_{m_i},x_{m_i+1})$$
$$= \varphi\left(max\left\{d(x_{n_i},x_{m_i}), d(x_{n_i},x_{n_i+1}), d(x_{m_i},x_{m_i+1}), \frac{d(x_{n_i},x_{m_i+1})+d(x_{m_i},x_{n_i+1})}{2}\right\}\right) + (K+1).\left[d(x_{n_i},x_{n_i+1}) + d(x_{m_i},x_{m_i+1})\right]$$

on letting $i \to \infty$, by using asymptotic regularity and upper semi-continuity of φ, we obtain

$$\epsilon = \lim_{i\to\infty} d(x_{n_i}, x_{m_i})$$
$$\leq \limsup_{i\to\infty} \varphi\left(max\left\{d(x_{n_i}, x_{m_i}), d(x_{n_i}, x_{n_i+1}), d(x_{m_i}, x_{m_i+1}), \frac{d(x_{n_i}, x_{m_i+1}) + d(x_{m_i}, x_{n_i+1})}{2}\right\}\right)$$
$$\leq \varphi(\epsilon) < \epsilon,$$

which is a contradiction.

Hence, $\{x_n\}$ is a Cauchy sequence.

Since X is complete, $\lim_{n\to\infty} x_n = z \in X$.

This z is a fixed point of T follow as in the proof of Theorem 2.2.

We now show the uniqueness of z.

Suppose y is another fixed point of T; $Ty = y, Tz = z$;

Suppose $y \neq z$.

$$d(Tz, Ty) \leq \varphi(M(z, y)) + K\{d(z, Tz) + d(y, Ty)\}$$
$$= \varphi\left(max\left\{d(z, y), d(z, Tz), d(y, Ty), \frac{d(z, Ty) + d(y, Tz)}{2}\right\}\right) + K\{d(z, z) + d(y, y)\}$$
$$= \varphi\left(max\left\{d(z, y), d(z, z), d(y, y), \frac{d(z, y) + d(y, z)}{2}\right\}\right)$$

$d(z, y) = d(Tz, Ty) \leq \varphi(d(z, y)) < d(z, y)$, a contradiction.

Therefore $z = y$ and z is the unique fixed point of T.

Hence, the theorem follows.

Example 4.4. Let $X = [0, \infty)$ with usual metric. Define $T: X \to X$ by $Tx = \begin{cases} 0, & if \ \ 0 \leq x < 1 \\ \frac{x}{x+1}, & if \ 1 \leq x < \infty \end{cases}$.

Let $\varphi \in \mathcal{U}$, and $\varphi(t) = \begin{cases} \frac{t}{3} & if \ 0 \leq t < 1 \\ \frac{1}{2}, & if \ 1 \leq t < \infty \end{cases}$.

Let $1 \leq x < \infty$. Then

$$Tx = \frac{x}{x+1}$$

$T^n x = \frac{x}{nx+1}$, $n = 1,2, \ldots$. Therefore

$$\lim_{n\to\infty} d(T^{n+1}x, T^n x) = \lim_{n\to\infty} \left|\frac{x}{(n+1)x + 1} - \frac{x}{nx + 1}\right|$$
$$= \lim_{n\to\infty} \left|\frac{1}{(n+1) + \frac{1}{x}} - \frac{1}{n + \frac{1}{x}}\right| = 0$$

Hence T is asymptotically regular.

Case I: Suppose $x \in [0,1)$ and $y \in [1, \infty)$

$$d(Tx, Ty) = \left|0 - \frac{y}{y+1}\right| = \frac{y}{y+1}$$
$$d(x, y) = |x - y| = y - x$$
$$d(x, Tx) = |x - 0| = x$$
$$d(y, Ty) = \left|y - \frac{y}{y+1}\right| = y - \frac{y}{y+1}$$
$$d(x, Ty) = \left|x - \frac{y}{y+1}\right| = \begin{cases} x - \frac{y}{y+1} & if \ x \geq \frac{y}{y+1} \\ \frac{y}{y+1} - x & if \ x < \frac{y}{y+1} \end{cases}$$
$$d(y, Tx) = |y - 0| = y$$

(i): Let us consider the case $x \geq \frac{y}{y+1}$ in this case,

$$M(x,y) = max\left\{y-x, x, y-\frac{y}{y+1}, \frac{x-\frac{y}{y+1}+y}{2}\right\}$$

$$= max\left\{y-x, x, y-\frac{y}{y+1}, \frac{1}{2}\left(x-\frac{y}{y+1}+y\right)\right\} \quad (4.4.1)$$

$M(x,y) = \frac{1}{2}\left(x-\frac{y}{y+1}+y\right)$ since $x < y$ and $x \geq \frac{y}{y+1}$

If $y-\frac{y}{y+1} \in [0,1)$ implies $M(x,y) = \frac{1}{3}\left(y-\frac{y}{y+1}\right)$, then

$$d(Tx,Ty) = \frac{y}{y+1} \leq y - \frac{y}{y+1}$$

$$\leq \frac{1}{3}\left(y-\frac{y}{y+1}\right) + 2\left(x+y-\frac{y}{y+1}\right)$$

$$= \varphi\big(M(x,y)\big) + K.\{d(x,Tx)+d(y,Ty)\} \text{ with } K = 2$$

If $y-\frac{y}{y+1} \in [1,\infty)$ implies $M(x,y) = \frac{1}{2}$, then

$$d(Tx,Ty) = \frac{y}{y+1} \leq y - \frac{y}{y+1}$$

$$\leq 2\left(x+y-\frac{y}{y+1}\right)$$

$$\leq \frac{1}{2} + 2\left(x+y-\frac{y}{y+1}\right)$$

$$= \varphi\big(M(x,y)\big) + K.\{d(x,Tx)+d(y,Ty)\} \text{ with } K = 2$$

Let us consider the case $x \leq \frac{y}{y+1}$ in this case,

$$M(x,y) = max\left\{y-x, x, y-\frac{y}{y+1}, \frac{\frac{y}{y+1}-x+y}{2}\right\}$$

$$= max\left\{y-x, x, y-\frac{y}{y+1}, \frac{1}{2}\left(\frac{y}{y+1}-x+y\right)\right\} \quad (4.4.2)$$

$M(x,y) = y - x$ since $x < y$ and $x \leq \frac{y}{y+1}$

If $y-x \in [0,1)$ implies $M(x,y) = \frac{1}{3}(y-x)$, then

$$d(Tx,Ty) = \frac{y}{y+1} \leq y - \frac{y}{y+1}$$

$$\leq x + y - \frac{y}{y+1}$$

$$\leq \frac{1}{3}(y-x) + 2\left(x+y-\frac{y}{y+1}\right)$$

$$= \varphi\big(M(x,y)\big) + K.\{d(x,Tx)+d(y,Ty)\} \text{ with } K = 2$$

If $y-x \in [1,\infty)$ implies $M(x,y) = \frac{1}{2}$, then

$$d(Tx,Ty) = \frac{y}{y+1} \leq y - \frac{y}{y+1}$$

$$\leq 2\left(x+y-\frac{y}{y+1}\right)$$

$$\leq \frac{1}{2} + 2\left(x+y-\frac{y}{y+1}\right)$$

$$= \varphi\big(M(x,y)\big) + K.\{d(x,Tx)+d(y,Ty)\} \text{ with } K = 2$$

Case II: Suppose $x, y \in [0,1)$

$d(Tx,Ty) = |0-0| = 0$

$d(x,y) = |x-y|$

$d(x,Tx) = |x - 0| = x$
$d(y,Ty) = |y - 0| = y$
$d(x,Ty) = |x - 0| = x$
$d(y,Tx) = |y - 0| = y$
So, clearly in this case, the inequality (4.4.1) holds.
Case III: Suppose $x, y \in [1,\infty)$ and $x < y \Rightarrow x - y < 0$

$$d(Tx,Ty) = \left|\frac{x}{x+1} - \frac{y}{y+1}\right| = \frac{y}{y+1} - \frac{x}{x+1}$$

$$d(x,y) = |x-y| = y - x$$

$$d(x,Tx) = \left|x - \frac{x}{x+1}\right| = x - \frac{x}{x+1}$$

$$d(y,Ty) = \left|y - \frac{y}{y+1}\right| = y - \frac{y}{y+1}$$

$$d(x,Ty) = \left|x - \frac{y}{y+1}\right| = x - \frac{y}{y+1}$$

$$d(y,Tx) = \left|y - \frac{x}{x+1}\right| = y - \frac{x}{x+1}$$

$$M(x,y) = max\left\{y - x, x - \frac{x}{x+1}, y - \frac{y}{y+1}, \frac{x - \frac{y}{y+1} + y - \frac{x}{x+1}}{2}\right\}$$

$$= max\left\{y - x, x - \frac{x}{x+1}, y - \frac{y}{y+1}, \frac{1}{2}\left(x - \frac{y}{y+1} + y - \frac{x}{x+1}\right)\right\}$$

Since $x, y \in [1,\infty)$ and $x < y$
$M(x,y) = \ y - \frac{y}{y+1}$
If $y - \frac{y}{y+1} \in [0,1)$ implies $M(x,y) = \frac{1}{3}\left(y - \frac{y}{y+1}\right)$, then

$$d(Tx,Ty) = \frac{y}{y+1} - \frac{x}{x+1} \le x - \frac{x}{x+1} + y - \frac{y}{y+1}$$

$$\le \frac{1}{3}\left(y - \frac{y}{y+1}\right) + 2\left(x - \frac{x}{x+1} + y - \frac{y}{y+1}\right)$$

$$= \varphi\big(M(x,y)\big) + K.\{d(x,Tx) + d(y,Ty)\} \text{ with } K = 2$$

If $y - \frac{y}{y+1} \in [1,\infty)$ implies $M(x,y) = \frac{1}{2}$, then

$$d(Tx,Ty) = \frac{y}{y+1} - \frac{x}{x+1} \le x - \frac{x}{x+1} + y - \frac{y}{y+1}$$

$$\le \frac{1}{2} + 2\left(x - \frac{x}{x+1} + y - \frac{y}{y+1}\right)$$

$$= \varphi\big(M(x,y)\big) + K.\{d(x,Tx) + d(y,Ty)\} \text{ with } K = 2$$

Case IV: Suppose $x, y \in [1,\infty)$ and $x > y \Rightarrow x - y > 0$

$$d(Tx,Ty) = \left|\frac{x}{x+1} - \frac{y}{y+1}\right| = \frac{x}{x+1} - \frac{y}{y+1}$$

$$d(x,y) = |x-y| = x - y$$

$$d(x,Tx) = \left|x - \frac{x}{x+1}\right| = x - \frac{x}{x+1}$$

$$d(y,Ty) = \left|y - \frac{y}{y+1}\right| = y - \frac{y}{y+1}$$

$$d(x,Ty) = \left|x - \frac{y}{y+1}\right| = x - \frac{y}{y+1}$$

$$d(y,Tx) = \left|y - \frac{x}{x+1}\right| = y - \frac{x}{x+1}$$

$$M(x,y) = max\left\{x-y, x-\frac{x}{x+1}, y-\frac{y}{y+1}, \frac{x-\frac{y}{y+1}+y-\frac{x}{x+1}}{2}\right\}$$

$$= max\left\{x-y, x-\frac{x}{x+1}, y-\frac{y}{y+1}, \frac{1}{2}\left(x-\frac{y}{y+1}+y-\frac{x}{x+1}\right)\right\} \text{ since } y<x.$$

$$M(x,y) = x-\frac{x}{x+1}$$

If $x-\frac{x}{x+1} \in [0,1)$ implies $M(x,y) = \frac{1}{3}\left(x-\frac{x}{x+1}\right)$, then

$$d(Tx,Ty) = \frac{x}{x+1} - \frac{y}{y+1} \leq x - \frac{x}{x+1} + y - \frac{y}{y+1}$$

$$\leq 2\left(x-\frac{x}{x+1}+y-\frac{y}{y+1}\right)$$

$$\leq \frac{1}{3}\left(x-\frac{x}{x+1}\right) + 2\left(x-\frac{x}{x+1}+y-\frac{y}{y+1}\right)$$

$$= \varphi\big(M(x,y)\big) + K.\{d(x,Tx)+d(y,Ty)\} \text{ with } K=2$$

If $x-\frac{x}{x+1} \in [1,\infty)$ implies $M(x,y) = \frac{1}{2}$, then

$$d(Tx,Ty) = \frac{x}{x+1} - \frac{y}{y+1} \leq x - \frac{x}{x+1} + y - \frac{y}{y+1}$$

$$\leq 2\left(x-\frac{x}{x+1}+y-\frac{y}{y+1}\right)$$

$$\leq \frac{1}{2} + 2\left(x-\frac{x}{x+1}+y-\frac{y}{y+1}\right)$$

$$= \varphi\big(M(x,y)\big) + K.\{d(x,Tx)+d(y,Ty)\} \text{ with } K=2$$

In all the cases T satisfies the hypothesis of Theorem 4.3. And clearly $'0'$ is the unique fixed point of T.

V. Conclusion

In conclusion, we have introduced and studied generalized Geraghty - Górnicki maps, generalized Boyd and Wong - Górnicki maps, and generalized weakly Górnicki maps in the context of complete metric spaces. Our work demonstrates the existence and uniqueness of fixed points for these maps, even in the absence of continuity. The results presented in this paper extend several known fixed point theorems, offering broader applicability in fixed point theory. The examples provided further illustrate the effectiveness and relevance of our generalizations.

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