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QUOTIENT-4 CORDIAL LABELING OF SOME BICYCLIC GRAPHS

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Abstract
Let G (V, E) be a simple graph of order p and size q. Let φ : V (G) → Z_5 – {0} be a function. For each edge set E (G) define the labeling φ^*: E (G) → Z_4 by $\varphi^*(uv) = \left\lceil \frac{\varphi(u)}{\varphi(v)} \right\rceil$ (mod 4) where $\varphi(u) \geq \varphi(v)$. The function φ is called Quotient-4 cordial labeling of G if $|v_\varphi(i) - v_\varphi(j)| \leq 1$, $1 \leq i, j \leq 4$, i≠j where $v_\varphi(x)$ denote the number of vertices labeled with x and $|e_\varphi(k) - e_\varphi(l)| \leq 1$, $0 \leq k, l \leq 3$, $k \neq l$, where $e_\varphi(y)$ denote the number of edges labeled with y. Here some bicyclic graphs such as $2C_r[P_s]$ and $By_{2,n}$ proved to be quotient-4 cordial graphs.

Keywords:
Cycle, bicycle, $2C_r[P_s]$ graph, Butterfly graph Quotient-4 cordial labeling, Quotient-4 cordial graph.

1. INTRODUCTION

Here the graphs considered are finite, simple, undirected and non-trivial. Graph theory has a good development in the graph labeling and has a broad range of applications. Refer Gallian [5] for more information. The cordial labeling concept was first introduced by Cahit [2]. H- and H2 –cordial labeling was introduced by Freeda S and Chellathurai R.S [3]. Mean Cordial Labeling was introduced by Albert William, Indira Rajasingh, and S Roy [1]. A graph G is said to be quotient-4 cordial graph if it receives quotient-4 cordial labeling .Let $v_\varphi(i)$ denotes the number of vertices labeled with i and $e_\varphi(k)$ denotes the number of edges labeled with k, $1 \leq i \leq 4$, $0 \leq k \leq 3$.

2. DEFINITIONS

Definition: 2.1[7] Let G (V, E) be a simple graph of order p and size q. Let φ: V (G) → Z_5 – {0} be a function. For each edge set E (G) define the labeling φ^*: E (G) → Z_4 by $\varphi^*(uv) = \left\lceil \frac{\varphi(u)}{\varphi(v)} \right\rceil$ (mod 4) where φ (u) ≥ φ (v). The function φ is called Quotient-4 cordial labeling of G if $|v_\varphi(i) - v_\varphi(j)| \leq 1$, $1 \leq i, j \leq 4$, i≠j where $v_\varphi(x)$ denote the number of vertices labeled with x and $|e_\varphi(k) - e_\varphi(l)| \leq 1$, $0 \leq k, l \leq 3$, $k \neq l$, where $e_\varphi(y)$ denote the number of edges labeled with y.

Definition: 2.2 Two cycles C_r of the same order connected with a path P_s yields a new graph and is denoted by $2C_r[P_s]$.

Definition: 2.3 Two cycles C_n of the same order sharing a common vertex with m pendent edges attached at the common vertex is called **butterfly graph** and it is denoted by $By_{m,n}$.

3. SOME TYPES OF BICYCLIC GRAPHS

Theorem: 3.1 A graph $2C_r[P_s]$ is quotient-4 cordial if $r > 2$ and $s > 1$.

Proof: Let G be a $2C_r[P_s]$ graph.

V (G) = $\{x_i, y_i, z_j : 1 \leq i \leq r, 2 \leq j \leq s-1\}$.

E (G) = $\{x_i x_{i+1}, y_i y_{i+1} : 1 \leq i \leq r-1\} \cup \{x_r x_1\} \cup \{y_r y_1\} \cup \{z_j z_{j+1} : 2 \leq j \leq s-2\} \cup \{x_1 z_2\} \cup \{z_{s-1} y_1\}$.

Here $|V(G)| = 2r + s - 2$, $|E(G)| = 2r + s - 1$.

Define φ : V (G) → {1, 2, 3, 4}.

The values of x_i are labeled as follows.

Case 1: When $r \equiv 0, 5$ (modulo 8) and $s \equiv 0, 1, 2, 3, 4, 5, 6, 7$ (modulo 8).
For $1 \leq i \leq r$.
φ (x_i) = 1 if i ≡ 4, 7 (modulo 8).
φ (x_i) = 2 if i ≡ 1, 2 (modulo 8).
φ (x_i) = 3 if i ≡ 5, 6 (modulo 8).
φ (x_i) = 4 if i ≡ 0, 3 (modulo 8).
Case 2: When $r \equiv 1$ (modulo 8) and $s \equiv 0, 1, 2, 3, 4, 7$ (modulo 8).
For $1 \leq i \leq r$, the labeling of x_i values are same as case 1.
Case 3: When $r \equiv 1$ (modulo 8) and $s \equiv 5$ (modulo 8).
For $1 \leq i \leq r - 1$, the labeling of x_i values are same as case 1.
φ (x_r) = 1.
Case 4: When $r \equiv 1$ (modulo 8) and $s \equiv 6$ (modulo 8).
For $1 \leq i \leq r - 1$, the labeling of x_i values are same as case 1.
φ (x_r) = 3.
Case 5: When $r \equiv 2$ (modulo 8) and $s \equiv 0$ (modulo 8).
For $1 \leq i \leq r - 2$, the labeling of x_i values are same as case 1.
φ (x_r) = 2, φ(x_{r-1}) = 1.
Case 6: When $r \equiv 2$ (modulo 8) and $s \equiv 1, 2, 3, 4, 5, 6$ (modulo 8).
For $1 \leq i \leq r - 2$, the labeling of x_i values are same as case 1.
φ (x_r) = 2, φ(x_{r-1}) = 3.
Case 7: When $r \equiv 2$ (modulo 8) and $s \equiv 7$ (modulo 8).
For $1 \leq i \leq r - 1$, the labeling of x_i values are same as case 1.
φ (x_r) = 3.
Case 8: When $r \equiv 3$ (modulo 8) and $s \equiv 0, 1, 2, 6, 7$ (modulo 8).
For $1 \leq i \leq r$, the labeling of x_i values are same as case 1.
Case 9: When $r, s = 3$.
φ (x_1) = 1, φ (x_2) = φ (x_3) = 2.
Case 10: When $r, s \equiv 3$ (modulo 8) and $r, s \neq 3$.
For $1 \leq i \leq r$, the labeling of x_i values are same as case 1.
Case 11: When $r = 3, s = 4$.
φ (x_1) = 3, φ (x_2) = φ (x_3) = 2.
Case 12: When $r \equiv 3$ (modulo 8), $s \equiv 4$ (modulo 8) and $r \neq 3, s \neq 4$.
For $1 \leq i \leq r$, the labeling of x_i values are same as case 1.
Case 13: When $r = 3, s = 5$.
φ (x_1) = φ (x_2) = φ (x_3) = 2.
Case 14: When $r \equiv 3$ (modulo 8), $s \equiv 5$ (modulo 8) and $r \neq 3, s \neq 5$.
For $1 \leq i \leq r$, the labeling of x_i values are same as case 1.
Case 15: When $r \equiv 4$ (modulo 8) and $s \equiv 0, 1$ (modulo 8).
For $1 \leq i \leq r - 1$, the labeling of x_i values are same as case 1.
φ (x_r) = 2.
Case 16: When $r \equiv 4$ (modulo 8) and $s \equiv 2$ (modulo 8).
For $1 \leq i \leq r - 4$, the labeling of x_i values are same as case 1.
φ (x_r) = 3, φ(x_{r-1}) = φ(x_{r-2}) = 2, φ(x_{r-3}) = 4.
Case 17: When $r \equiv 4$ (modulo 8) and $s \equiv 3, 4, 7$ (modulo 8).
For $1 \leq i \leq r - 2$, the labeling of x_i values are same as case 1.
φ (x_r) = 1, φ(x_{r-1}) = 3.
Case 18: When $r \equiv 4$ (modulo 8) and $s \equiv 5, 6$ (modulo 8).

For $1 \leq i \leq r-1$, the labeling of x_i values are same as case 1.
φ (x_r) = 4.
Case 19: When $r \equiv 6$ (modulo 8) and $s \equiv 0, 5, 7$ (modulo 8).
For $1 \leq i \leq r$, the labeling of x_i values are same as case 1.
Case 20: When $r \equiv 6$ (modulo 8) and $s \equiv 1, 2, 6$ (modulo 8).
For $1 \leq i \leq r-1$, the labeling of x_i values are same as case 1.
φ (x_r) = 2.
Case 21: When $r \equiv 6$ (modulo 8) and $s \equiv 3, 4$ (modulo 8).
For $1 \leq i \leq r-1$, the labeling of x_i values are same as case 1.
φ(x_r) = 1.
Case 22: When $r \equiv 7$ (modulo 8) and $s \equiv 0, 3, 7$ (modulo 8).
For $1 \leq i \leq r-1$, the labeling of x_i values are same as case 1.
φ (x_r) = 2.
Case 23: When $r \equiv 7$ (modulo 8) and $s \equiv 1, 2$ (modulo 8).
For $1 \leq i \leq r-2$, the labeling of x_i values are same as case 1.
φ (x_r) = 2, φ (x_{r-1}) = 1.
Case 24: When $r \equiv 7$ (modulo 8) and $s \equiv 4$ (modulo 8).
For $1 \leq i \leq r-2$, the labeling of x_i values are same as case 1.
φ (x_r) = 3, φ (x_{r-1}) = 1.
Case 25: When $r \equiv 7$ (modulo 8) and $s \equiv 5, 6$ (modulo 8).
For $1 \leq i \leq r$, the labeling of x_i values are same as case 1.

The values of y_i are labeled as follows.
Case 1: When $r \equiv 0$ (modulo 8) and $s \equiv 0, 1, 2, 3, 4, 5, 6, 7$ (modulo 8).
For $1 \leq i \leq r$.
φ (y_i) = 1 if i ≡ 2, 5 (modulo 8).
φ (y_i) = 2 if i ≡ 0, 7 (modulo 8).
φ (y_i) = 3 if i ≡ 1, 6 (modulo 8).
φ (y_i) = 4 if i ≡ 3, 4 (modulo 8).
Case 2: When $r \equiv 1$ (modulo 8) and $s \equiv 0, 4, 5$ (modulo 8).
For $1 \leq i \leq r$, the labeling of y_i values are same as case 1.
Case 3: When $r \equiv 1$ (modulo 8) and $s \equiv 1, 2, 6$ (modulo 8).
For $1 \leq i \leq r-1$, the labeling of y_i values are same as case 1.
φ (y_r) = 1.
Case 4: When $r \equiv 1$ (modulo 8) and $s \equiv 3$ (modulo 8).
For $1 \leq i \leq r-2$, the labeling of y_i values are same as case 1.
φ (y_r) = 1, φ (y_{r-1}) = 4.
Case 5: When $r \equiv 1$ (modulo 8) and $s \equiv 7$ (modulo 8).
For $1 \leq i \leq r-1$, the labeling of y_i values are same as case 1.
φ(y_r) = 4.
Case 6: When $r \equiv 2$ (modulo 8) and $s \equiv 0$ (modulo 8).
For $1 \leq i \leq r-1$, the labeling of y_i values are same as case 1.
φ (y_r) = 4.
Case 7: When $r \equiv 2$ (modulo 8) and $s \equiv 1, 2, 3, 4, 5, 6, 7$ (modulo 8).
For $1 \leq i \leq r-2$, the labeling of y_i values are same as case 1.
φ (y_r) = 1, φ (y_{r-1}) = 4.
Case 8: When $r \equiv 3$ (modulo 8) and $s \equiv 0, 1$ (modulo 8).
For $1 \leq i \leq r$.

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φ (y_i) = 1 if i ≡ 2, 5 (modulo 8).
φ (y_i) = 2 if i ≡ 0, 7 (modulo 8).
φ (y_i) = 3 if i ≡ 3, 4 (modulo 8).
φ (y_i) = 4 if i ≡ 1, 6 (modulo 8).

Case 9: When $r = 3, s = 2$.
φ (y_1) = φ (y_3) = 1, φ (y_2) = 2.

Case 10: When $r \equiv 3$ (modulo 8), $s \equiv 2$ (modulo 8) and $r \neq 3, s \neq 2$.
For $1 \leq i \leq r - 1$, the labeling of y_i values are same as case 8.
φ (y_r) = 4.

Case 11: When $r = 3, s = 3$.
φ (y_1) = 1, φ (y_2) = φ (y_3) = 3.

Case 12: When $r, s \equiv 3$ (modulo 8) and $r, s \neq 3$.
For $1 \leq i \leq r - 3$, the labeling of y_i values are same as case 8.
φ (y_r) = 1, φ (y_{r-1}) = 3, φ (y_{r-2}) = 2.

Case 13: When $r = 3, s = 4, 6$.
φ (y_1) = φ (y_3) = 4, φ (y_2) = 1.

Case 14: When $r \equiv 3$ (modulo 8), $s \equiv 4$ (modulo 8) and $r \neq 3, s \neq 4$.
For $1 \leq i \leq r - 3$, the labeling of y_i values are same as case 8.
φ (y_r) = 3, φ (y_{r-1}) = 4, φ (y_{r-2}) = 2.

Case 15: When $r \equiv 3$ (modulo 8) and $s \equiv 5$ (modulo 8).
For $1 \leq i \leq r - 3$, the labeling of y_i values are same as case 8.
φ (y_r) = 4, φ (y_{r-1}) = 1, φ (y_{r-2}) = 3.

Case 16: When $r \equiv 3$ (modulo 8), $s \equiv 6$ (modulo 8) and $r \neq 3, s \neq 6$.
For $1 \leq i \leq r - 4$, the labeling of y_i values are same as case 8.
φ (y_r) = φ (y_{r-3}) = 1, φ (y_{r-1}) = φ (y_{r-2}) = 3.

Case 17: When $r = 3, s = 7$.
φ (y_1) = φ (y_3) = 3, φ (y_2) = 2.

Case 18: When $r \equiv 3$ (modulo 8), $s \equiv 7$ (modulo 8) and $r \neq 3, s \neq 7$.
For $1 \leq i \leq r$, the labeling of y_i values are same as case 15.

Case 19: When $r \equiv 4$ (modulo 8) and $s \equiv 0, 1, 2, 3, 4, 5, 6, 7$ (modulo 8).
For $1 \leq i \leq r - 1$, the labeling of y_i values are same as case 8.
φ (y_r) = 1.

Case 20: When $r \equiv 5$ (modulo 8) and $s \equiv 0, 1$ (modulo 8).
For $1 \leq i \leq r - 2$, the labeling of y_i values are same as case 1.
φ (y_r) = 3, φ (y_{r-1}) = 2.

Case 21: When $r \equiv 5$ (modulo 8) and $s \equiv 2, 3, 4, 5, 6, 7$ (modulo 8).
For $1 \leq i \leq r$, the labeling of y_i values are same as case 1.

Case 22: When $r \equiv 6$ (modulo 8) and $s \equiv 0, 7$ (modulo 8).
For $1 \leq i \leq r - 1$, the labeling of y_i values are same as case 1.
φ (y_r) = 2.

Case 23: When $r \equiv 6$ (modulo 8) and $s \equiv 1, 2, 3, 4$ (modulo 8).
For $1 \leq i \leq r$, the labeling of y_i values are same as case 1.

Case 24: When $r \equiv 6$ (modulo 8) and $s \equiv 5, 6$ (modulo 8).
For $1 \leq i \leq r - 1$, the labeling of y_i values are same as case 1.
φ (y_r) = 4.

Case 25: When $r \equiv 7$ (modulo 8) and $s \equiv 0, 1, 5, 6, 7$ (modulo 8).
For $1 \leq i \leq r$, the labeling of y_i values are same as case 1.

Case 26: When $r \equiv 7$ (modulo 8) and $s \equiv 2$ (modulo 8).
For $1 \leq i \leq r-2$, the labeling of y_i values are same as case 1.
$\varphi(y_r) = 3$, $\varphi(y_{r-1}) = 4$.
Case 27: When $r \equiv 7$ (modulo 8) and $s \equiv 3$ (modulo 8).
For $1 \leq i \leq r-4$, the labeling of y_i values are same as case 1.
$\varphi(y_r) = \varphi(y_{r-3}) = 1$, $\varphi(y_{r-1}) = 3$, $\varphi(y_{r-2}) = 4$.
Case 28: When $r \equiv 7$ (modulo 8) and $s \equiv 4$ (modulo 8).
For $1 \leq i \leq r-2$, the labeling of y_i values are same as case 1.
$\varphi(y_r) = 2$, $\varphi(y_{r-1}) = 4$.

The values of z_j are labeled as follows.
Case 1: When $r \equiv 0$ (modulo 8) and $s \equiv 1, 2, 3, 5, 6$ (modulo 8).
For $2 \leq j \leq s-1$.
$\varphi(z_j) = 1$ if $j \equiv 4, 7$ (modulo 8).
$\varphi(z_j) = 2$ if $j \equiv 1, 2$ (modulo 8).
$\varphi(z_j) = 3$ if $j \equiv 5, 6$ (modulo 8).
$\varphi(z_j) = 4$ if $j \equiv 0, 3$ (modulo 8).
Case 2: When $r \equiv 0$ (modulo 8) and $s \equiv 0$ (modulo 8).
For $2 \leq j \leq s-3$, the labeling of z_j values are same as case 1.
$\varphi(z_{s-1}) = 1$, $\varphi(z_{s-2}) = 4$.
Case 3: When $r \equiv 0$ (modulo 8) and $s \equiv 4$ (modulo 8).
For $2 \leq j \leq s-2$, the labeling of z_j values are same as case 1.
$\varphi(z_{s-1}) = 1$.
Case 4: When $r \equiv 0$ (modulo 8) and $s \equiv 7$ (modulo 8).
For $2 \leq j \leq s-5$, the labeling of z_j values are same as case 1.
$\varphi(z_{s-1}) = \varphi(z_{s-4}) = 1$, $\varphi(z_{s-2}) = 3$, $\varphi(z_{s-3}) = 4$.
Case 5: When $r \equiv 1$ (modulo 8) and $s \equiv 0$ (modulo 8).
For $2 \leq j \leq s-3$, the labeling of z_j values are same as case 1.
$\varphi(z_{s-1}) = 4$, $\varphi(z_{s-2}) = 1$.
Case 6: When $r \equiv 1$ (modulo 8) and $s \equiv 1, 2, 3, 5, 6$ (modulo 8).
For $2 \leq j \leq s-1$, the labeling of z_j values are same as case 1.
Case 7: When $r \equiv 1$ (modulo 8) and $s \equiv 4$ (modulo 8).
For $2 \leq j \leq s-3$, the labeling of z_j values are same as case 1.
$\varphi(z_{s-1}) = 1$, $\varphi(z_{s-2}) = 4$.
Case 8: When $r \equiv 1$ (modulo 8) and $s \equiv 7$ (modulo 8).
For $2 \leq j \leq s-5$, the labeling of z_j values are same as case 1.
$\varphi(z_{s-1}) = \varphi(z_{s-3}) = 1$, $\varphi(z_{s-2}) = 4$, $\varphi(z_{s-4}) = 3$.
Case 9: When $r \equiv 2$ (modulo 8) and $s \equiv 0, 1, 2, 3, 5, 6, 7$ (modulo 8).
For $2 \leq j \leq s-1$, the labeling of z_j values are same as case 1.
Case 10: When $r \equiv 2$ (modulo 8) and $s \equiv 4$ (modulo 8).
$\varphi(z_{s-1}) = 1$.
For $2 \leq j \leq s-2$, the labeling of z_j values are same as case 1.
Case 11: When $r \equiv 3$ (modulo 8) and $s \equiv 0, 1$ (modulo 8).
For $2 \leq j \leq s-1$, the labeling of z_j values are same as case 1.
Case 12: When $r \equiv 3$ (modulo 8) and $s \equiv 2, 5$ (modulo 8).
For $2 \leq j \leq s-3$, the labeling of z_j values are same as case 1.

φ (z_{s-1}) = 1, φ (z_{s-2}) = 3.
Case 13: When $r, s = 3$.
φ (z_2) = 4.
Case 14: When $r, s \equiv 3$ (modulo 8) and $r, s \neq 3$.
For $2 \leq j \leq s - 3$, the labeling of z_j values are same as case 1.
φ (z_{s-1}) = 3, φ (z_{s-2}) = 1.
Case 15: When $r = 3, s = 4$.
φ (z_2) = 1, φ (z_3) = 3 .
Case 16: When $r \equiv 3$ (modulo 8), $s \equiv 4$ (modulo 8) and $r \neq 3, s \neq 4$.
For $2 \leq j \leq s - 4$, the labeling of z_j values are same as case 1.
φ (z_{s-1}) = φ (z_{s-3}) = 1, φ (z_{s-2}) = 3.
Case 17: When $r = 3, s = 5$.
φ (z_2) = 3, φ (z_3) = 1, φ (z_4) = 4 .
Case 18: When $r \equiv 3$ (modulo 8), $s \equiv 5$ (modulo 8) and $r \neq 3, s \neq 5$.
For $2 \leq j \leq s - 1$, the labeling of z_j values are same as case 12.
Case 19: When $r = 3, s = 6$.
φ (z_2) = φ (z_5) = 1, φ (z_3) = φ (z_4) = 3.
Case 20: When $r \equiv 3$ (modulo 8), $s \equiv 6$ (modulo 8) and $r \neq 3, s \neq 6$.
For $2 \leq j \leq s - 2$, the labeling of z_j values are same as case 1.
φ (z_{s-1}) = 4.
Case 21: When $r = 3, s = 7$.
φ (z_2) = 2, φ (z_3) = φ (z_5) = 4, φ (z_4) = φ (z_6) = 1.
Case 22: When $r \equiv 3$ (modulo 8), $s \equiv 7$ (modulo 8) and $r \neq 3, s \neq 7$.
For $2 \leq j \leq s - 2$, the labeling of z_j values are same as case 1.
φ (z_{s-1}) = 1.
Case 23: When $r \equiv 4$ (modulo 8) and $s \equiv 2, 5, 7$ (modulo 8).
For $2 \leq j \leq s - 1$.
φ (z_j) = 1 if j ≡ 4, 7 (modulo 8).
φ (z_j) = 2 if j ≡ 1, 2 (modulo 8).
φ (z_j) = 3 if j ≡ 0, 3 (modulo 8).
φ (z_j) = 4 if j ≡ 5, 6 (modulo 8).
Case 24: When $r \equiv 4$ (modulo 8) and $s \equiv 0, 1, 6$ (modulo 8).
For $2 \leq j \leq s - 3$, the labeling of z_j values are same as case 23.
φ (z_{s-1}) = 1, φ (z_{s-2}) = 3.
Case 25: When $r \equiv 4$ (modulo 8) and $s \equiv 3, 4$ (modulo 8).
For $2 \leq j \leq s - 2$, the labeling of z_j values are same as case 23.
φ (z_{s-1}) = 4.
Case 26: When $r \equiv 5$ (modulo 8) and $s \equiv 0, 1, 2, 3, 4, 6$ (modulo 8).
For $2 \leq j \leq s - 1$, the labeling of z_j values are same as case 23.
Case 27: When $r \equiv 5$ (modulo 8) and $s \equiv 5$ (modulo 8).
For $2 \leq j \leq s - 3$, the labeling of z_j values are same as case 23.
φ (z_{s-1}) = 3, φ (z_{s-2}) = 1.
Case 28: When $r \equiv 5$ (modulo 8) and $s \equiv 7$ (modulo 8).
For $2 \leq j \leq s - 2$, the labeling of z_j values are same as case 23.
φ (z_{s-1}) = 3.
Case 29: When $r \equiv 6$ (modulo 8) and $s \equiv 0, 2, 3, 5$ (modulo 8).

For $2 \leq j \leq s-1$, the labeling of z_j values are same as case 23.
Case 30: When $r \equiv 6$ (modulo 8) and $s \equiv 1$ (modulo 8).
For $2 \leq j \leq s-4$, the labeling of z_j values are same as case 23.
φ (z_{s-1}) = 4, φ (z_{s-2}) = 1, φ (z_{s-3}) = 3.
Case 31: When $r \equiv 6$ (modulo 8) and $s \equiv 4$ (modulo 8).
For $2 \leq j \leq s-2$, the labeling of z_j values are same as case 23.
φ (z_{s-1}) = 4.
Case 32: When $r \equiv 6$ (modulo 8) and $s \equiv 6$ (modulo 8).
For $2 \leq j \leq s-2$, the labeling of z_j values are same as case 17.
φ (z_{s-1}) = 3.
Case 33: When $r \equiv 6, 7$ (modulo 8) and $s \equiv 7$ (modulo 8).
For $2 \leq j \leq s-2$, the labeling of z_j values are same as case 23.
φ (z_{s-1}) = 1.
Case 34: When $r \equiv 7$ (modulo 8) and $s \equiv 0, 1, 2, 3, 4$ (modulo 8).
For $2 \leq j \leq s-1$, the labeling of z_j values are same as case 23.
Case 35: When $r \equiv 7$ (modulo 8) and $s \equiv 5$ (modulo 8).
For $2 \leq j \leq s-3$, the labeling of z_j values are same as case 23.
φ (z_{s-1}) = 1, φ (z_{s-2}) = 4.
Case 36: When $r \equiv 7$ (modulo 8) and $s \equiv 6$ (modulo 8).
For $2 \leq j \leq s-4$, the labeling of z_j values are same as case 23.
φ (z_{s-1}) = 3, φ (z_{s-2}) = 1, φ (z_{s-3}) = 4.
The following table concurrence is realized with modulo value 8.

Nature of r and s	$v_\varphi(1)$	$v_\varphi(2)$	$v_\varphi(3)$	$v_\varphi(4)$
r ≡ 0 s ≡ 0	$\frac{2r+s}{4}$	$\frac{2r+s}{4}-1$	$\frac{2r+s}{4}-1$	$\frac{2r+s}{4}$
r ≡ 0 s ≡ 1	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}-1$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$
r ≡ 0 s ≡ 2, 6	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$
r ≡ 0 s ≡ 3	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$
r ≡ 0 s ≡ 4	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}-1$	$\frac{2r+s}{4}-1$
r ≡ 0 s ≡ 5	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}-1$	$\frac{2r+s-1}{4}$
r ≡ 0 s ≡ 7	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$
r ≡ 1 s ≡ 0, 4	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$
r ≡ 1 s ≡ 1, 5	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$
r ≡ 1 s ≡ 2	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}-1$	$\frac{2r+s}{4}-1$
r ≡ 1 s ≡ 3, 7	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}-1$	$\frac{2r+s-1}{4}$

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$r \equiv 1$ $s \equiv 6$	$\frac{2r+s}{4}$	$\frac{2r+s}{4} - 1$	$\frac{2r+s}{4}$	$\frac{2r+s}{4} - 1$
$r \equiv 2$ $s \equiv 0$	$\frac{2r+s}{4}$	$\frac{2r+s}{4} - 1$	$\frac{2r+s}{4}$	$\frac{2r+s}{4} - 1$
$r \equiv 2$ $s \equiv 1$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4} - 1$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$
$r \equiv 2$ $s \equiv 2, 6$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$
$r \equiv 2$ $s \equiv 3$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4} + 1$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$
$r \equiv 2$ $s \equiv 4$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$	$\frac{2r+s}{4} - 1$	$\frac{2r+s}{4} - 1$
$r \equiv 2$ $s \equiv 5$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4} - 1$	$\frac{2r+s-1}{4}$
$r \equiv 2$ $s \equiv 7$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4} + 1$	$\frac{2r+s-3}{4}$
$r \equiv 3$ $s \equiv 0, 4$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$
$r \equiv 3$ $s \equiv 1$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4} + 1$
$r \equiv 3$ $s \equiv 2, 6$	$\frac{2r+s}{4}$	$\frac{2r+s}{4} - 1$	$\frac{2r+s}{4} - 1$	$\frac{2r+s}{4}$
$r \equiv 3$ $s \equiv 3$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4} - 1$
$r \equiv 3$ $s \equiv 5$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4} + 1$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$
$r \equiv 3$ $s \equiv 7$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4} - 1$	$\frac{2r+s-1}{4}$
$r \equiv 4$ $s \equiv 0, 4$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$	$\frac{2r+s}{4} - 1$	$\frac{2r+s}{4} - 1$
$r \equiv 4$ $s \equiv 1, 5$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4} - 1$	$\frac{2r+s-1}{4}$
$r \equiv 4$ $s \equiv 2, 6$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$
$r \equiv 4$ $s \equiv 3, 7$	$\frac{2r+s-3}{4} + 1$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$
$r \equiv 5$ $s \equiv 0, 4$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$
$r \equiv 5$ $s \equiv 1$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4} + 1$	$\frac{2r+s-3}{4}$
$r \equiv 5$ $s \equiv 2, 6$	$\frac{2r+s}{4}$	$\frac{2r+s}{4} - 1$	$\frac{2r+s}{4} - 1$	$\frac{2r+s}{4}$
$r \equiv 5$ $s \equiv 3$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4} - 1$	$\frac{2r+s-1}{4}$
$r \equiv 5$ $s \equiv 5$	$\frac{2r+s-3}{4} + 1$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$

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$r \equiv 5$ $s \equiv 7$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}-1$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$
$r \equiv 6$ $s \equiv 0, 4$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}-1$	$\frac{2r+s}{4}-1$	$\frac{2r+s}{4}$
$r \equiv 6$ $s \equiv 1, 5$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}-1$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$
$r \equiv 6$ $s \equiv 2, 6$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$
$r \equiv 6$ $s \equiv 3, 7$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$
$r \equiv 7$ $s \equiv 0, 4$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$
$r \equiv 7$ $s \equiv 1, 5$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$
$r \equiv 7$ $s \equiv 2$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}-1$	$\frac{2r+s}{4}-1$	$\frac{2r+s}{4}$
$r \equiv 7$ $s \equiv 3, 7$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}-1$
$r \equiv 7$ $s \equiv 6$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}-1$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}-1$

Table 1: Vertex labeling of $2C_r\,[P_s]$ graph.

The following table concurrence is realized with modulo value 8.

Nature of r and s	$e_\varphi(0)$	$e_\varphi(1)$	$e_\varphi(2)$	$e_\varphi(3)$
$r \equiv 0$ $s \equiv 0$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}-1$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$
$r \equiv 0$ $s \equiv 1, 5$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$
$r \equiv 0$ $s \equiv 2$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}+1$	$\frac{2r+s-2}{4}$
$r \equiv 0$ $s \equiv 3$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}$
$r \equiv 0$ $s \equiv 4$	$\frac{2r+s}{4}-1$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$
$r \equiv 0$ $s \equiv 6$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}+1$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$
$r \equiv 0$ $s \equiv 7$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}+1$
$r \equiv 1$ $s \equiv 0, 4$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}+1$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$
$r \equiv 1$ $s \equiv 1$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}+1$
$r \equiv 1$ $s \equiv 2, 6$	$\frac{2r+s}{4}-1$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$
$r \equiv 1$ $s \equiv 3, 7$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$

r ≡ 1 s ≡ 5	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$
r ≡ 2 s ≡ 0, 4	$\frac{2r+s}{4}-1$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$
r ≡ 2 s ≡ 1, 5	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$
r ≡ 2 s ≡ 2	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}+1$	$\frac{2r+s-2}{4}$
r ≡ 2 s ≡ 3, 7	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}$
r ≡ 2 s ≡ 6	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}+1$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$
r ≡ 3 s ≡ 0, 4	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}+1$	$\frac{2r+s-2}{4}$
r ≡ 3 s ≡ 1, 5	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}$
r = 3 s = 2	1	2	2	2
r ≡ 3 and $r \neq$ 3 s ≡ 2 and $s \neq$ 2	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}-1$	$\frac{2r+s}{4}$
r ≡ 3 s ≡ 3, 7	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$
r ≡ 3 s ≡ 6	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}-1$
r ≡ 4 s ≡ 0	$\frac{2r+s}{4}$	$\frac{2r+s}{4}-1$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$
r ≡ 4 s ≡ 1, 5	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$
r ≡ 4 s ≡ 2	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}+1$	$\frac{2r+s-2}{4}$
r ≡ 4 s ≡ 3	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}+1$
r ≡ 4 s ≡ 4	$\frac{2r+s}{4}-1$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$
r ≡ 4 s ≡ 6	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}+1$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$
r ≡ 4 s ≡ 7	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}+1$
r ≡ 5 s ≡ 0	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}+1$	$\frac{2r+s-2}{4}$
r ≡ 5 s ≡ 1	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}$
r ≡ 5 s ≡ 2, 6	$\frac{2r+s}{4}$	$\frac{2r+s}{4}-1$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$

r ≡ 5 s ≡ 3, 7	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$
r ≡ 5 s ≡ 4	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}+1$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$
r ≡ 5 s ≡ 5	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}+1$
r ≡ 6 s ≡ 0	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}-1$
r ≡ 6 s ≡ 1, 5	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$
r ≡ 6 s ≡ 2, 6	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}+1$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$
r ≡ 6 s ≡ 3	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}+1$
r ≡ 6 s ≡ 4	$\frac{2r+s}{4}-1$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$
r ≡ 6 s ≡ 7	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}$
r ≡ 7 s ≡ 0	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}+1$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$
r ≡ 7 s ≡ 1	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}+1$
r ≡ 7 s ≡ 2	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}-1$
r ≡ 7 s ≡ 3, 7	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$	$\frac{2r+s-1}{4}$
r ≡ 7 s ≡ 4	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}$	$\frac{2r+s-2}{4}+1$	$\frac{2r+s-2}{4}$
r ≡ 7 s ≡ 5	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}$	$\frac{2r+s-3}{4}+1$	$\frac{2r+s-3}{4}+1$
r ≡ 7 s ≡ 6	$\frac{2r+s}{4}-1$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$	$\frac{2r+s}{4}$

Table 2: Edge labeling of $2C_r[P_s]$ graph.

The above tables 1 and 2 we find that $|v_\varphi(i) - v_\varphi(j)| \le 1$ and $|e_\varphi(k) - e_\varphi(l)| \le 1$. Hence $2C_r[P_s]$ is quotient-4 cordial labeling.

Theorem: 3.2 A type of butterfly graph $By_{2,n}$ is quotient-4 cordial if $n \ge 3$.
Proof: Let G = $By_{2,n}$ be a butterfly graph.
V (G) = $\{x_i, y_j, z_k : 1 \le i \le n, 1 \le j \le n-1, 1 \le k \le 2\}$.
E (G) = $\{x_i x_{i+1}: 1 \le i \le n-1\} \cup \{x_1 x_n\} \cup \{(y_j y_{j+1}) : 1 \le j \le n-2\} \cup \{x_1 y_1\} \cup \{x_1 y_{n-1}\} \cup \{x_1 z_1\} \cup \{x_1 z_2\}$.
Here $|V(G)| = 2n+1$, $|E(G)| = 2n+2$.
Define φ : V (G) → {1, 2, 3, 4}.

The values of x_i are labeled as follows.
Case 1: When $n \equiv 0, 2, 6$ (modulo 8).
For $1 \le i \le n$.

φ (x_i) = 1 if i ≡ 1, 6 (modulo 8).
φ (x_i) = 2 if i ≡ 3, 4 (modulo 8).
φ (x_i) = 3 if i ≡ 2, 5 (modulo 8).
φ (x_i) = 4 if i ≡ 0, 7 (modulo 8).
Case 2: When $n \equiv 1, 7$ (modulo 8).
For $1 \leq i \leq n-2$, the labeling of x_i values are same as case 1.
φ (x_n) = 1, φ (x_{n-1}) = 3.
Case 3: When $n \equiv 3$ (modulo 8).
For $1 \leq i \leq n-1$, the labeling of x_i values are same as case 1.
φ (x_n) = 3.
Case 4: When $n \equiv 4$ (modulo 8).
For $1 \leq i \leq n-2$, the labeling of x_i values are same as case 1.
φ (x_n) = 4, φ (x_{n-1}) = 1.
Case 5: When $n \equiv 5$ (modulo 8).
For $1 \leq i \leq n-3$, the labeling of x_i values are same as case 1.
φ (x_n) = φ (x_{n-1}) = 1, φ (x_{n-2}) = 4.

The values of y_j are labeled as follows.
Case 1: When $n \equiv 0, 1, 2, 3, 4, 5, 6, 7$ (modulo 8).
For $1 \leq i \leq n-1$.
φ (y_j) = 1 if j ≡ 0, 5 (modulo 8).
φ (y_j) = 2 if j ≡ 2, 3 (modulo 8).
φ (y_j) = 3 if j ≡ 6, 7 (modulo 8).
φ (y_j) = 4 if j ≡ 1, 4 (modulo 8).
The values of z_k are labeled as follows.
Case 1: When $n \equiv 0, 3$ (modulo 8).
φ (z_1) = 1, φ (z_2) = 4.
Case 2: When $n \equiv 1, 7$ (modulo 8).
φ (z_1) = φ (z_2) = 4.
Case 3: When $n \equiv 2$ (modulo 8).
φ (z_1) = 1, φ (z_2) = 2.
Case 4: When $n \equiv 4$ (modulo 8).
φ (z_1) = 1, φ (z_2) = 3.
Case 5: When $n \equiv 5$ (modulo 8).
φ (z_1) = φ (z_2) = 3.
Case 6: When $n \equiv 6$ (modulo 8).
φ (z_1) = 3, φ (z_2) = 4.
The following table concurrence is realized with modulo value 8.

Nature of n	$v_\varphi(1)$	$v_\varphi(2)$	$v_\varphi(3)$	$v_\varphi(4)$
n ≡ 0	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}+1$
n ≡ 1, 3, 5	$\frac{n+1}{2}$	$\frac{n+1}{2}-1$	$\frac{n+1}{2}$	$\frac{n+1}{2}$
n ≡ 2, 4	$\frac{n}{2}+1$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$
n ≡ 6	$\frac{n}{2}$	$\frac{n}{2}+1$	$\frac{n}{2}$	$\frac{n}{2}$

n ≡ 7	$\frac{n+1}{2}-1$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$

Table 7: Vertex labeling of $By_{2,n}$ graph.

The following table concurrence is realized with modulo value 8.

Nature of n	$e_\varphi(0)$	$e_\varphi(1)$	$e_\varphi(2)$	$e_\varphi(3)$
n ≡ 0	$\frac{n}{2}+1$	$\frac{n}{2}+1$	$\frac{n}{2}$	$\frac{n}{2}$
n ≡ 1, 3, 5, 7	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$
n ≡ 2, 4	$\frac{n}{2}+1$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}+1$
n ≡ 6	$\frac{n}{2}$	$\frac{n}{2}+1$	$\frac{n}{2}+1$	$\frac{n}{2}$

Table 8: Edge labeling of $By_{2,n}$ graph.

The above tables 7 and 8 we find that $|v_\varphi(i) - v_\varphi(j)| \leq 1$ and $|e_\varphi(k) - e_\varphi(l)| \leq 1$. Hence $By_{2,n}$ is quotient-4 cordial labeling.

4.CONCLUSION

In this paper, it is proved that some bicyclic graphs which admits quotient-4 cordial. The existence of quotient-4 cordial labeling of different families of graphs will be the future work.

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