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FOURIER TRANSFORM AND MATHEMATICS: A REVIEW

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Abstract
The Fourier Transformation is a mathematical technique that decomposes a function into its constituent frequencies, revolutionizing variegated fields like mathematics. It originated from the work of a French Mathematician Jean-Baptiste Joseph Fourier (1768-1830) and has since then became a cornerstone of mathematical analysis. Fourier transforms are useful in dealing with various problems in the field of mathematics, Physics and Engineering. This transformation is simplest, time saving and important in the present era of technology. It has its role in power distribution system, mechanical system, industries and wireless networks etc. The said transformation helps in the solution of integral equations, differential equations and aids in understanding of complex functions. The paper aims to delve into fundamental concepts of Fourier transformation, its mathematical properties and application in Mathematics
Key Words: Fourier Transform, mathematics, integral equations, differential equations, functions.

Introduction

The Fourier Transform is a mathematical technique proposed by a French Mathematician Jean-Baptiste Joseph Fourier (1768-1830), used to present a time domain signal as a sum of sine and cosine waves in the frequency domain. It is one of the most important mathematical tools in a wide variety of fields in science and engineering. This transformation is simple, time saving and important in real life usage in the recent era of machine learning and artificial intelligence. The Fourier transform also relates to topics of linear algebra, like the expression of a vector as linear combinations of an orthonormal basis, or expressed as linear combinations of eigenvectors of a matrix (or a linear operator). In the arena of Mathematics, these transforms are supportive in the solution of integral equations and differential equations that is discussed in this paper. Fourier transforms are generally of two types namely finite and infinity depending on the limits of variable used. Many research scholars have thrown light from their perspectives on the application of Fourier transforms in different in the present era of science and technology. Anupama Gupta (2013) has discussed Fourier Transform and Its Application in Cell Phones, Nathan Lenssen, D. N. (2014) studied an Introduction to Fourier Analysis with applications to Music. Kalyani Hande, P. F. (2015) explained applications of Fourier series in communication system. S. Shenbaga Ezhil (2017) elaborated real time application of Fourier Transforms. Poonam Rajvardhan &Shankar Akram Patil (2018) examined application of Fourier Transform in Engineering and technology and so on. After review of previous literature ,I tried to study about role of Fourier transform in the branch of Mathematics.

Objective of the study

Fourier Transform is applicable in various fields of science and technology as well as in real life also. Here I have tried to analyse how Fourier Transform is supportive in the area of Mathematics by taking examples.

Research Methodology:

By proper study of literature available/research papers, example, reviewed latest research done on this topic.

Basic Definitions

In the beginning, I throw light on some elementary definitions and properties of the Fourier transform that are basis of the paper to proceed further.

4.1 Periodic Function: A function f : R → R is said to be periodic if f(x + ⅄) = f(⅄) for all x € R, (⅄ ≠ 0).The smallest value of ⅄ is named the period.

4.2 Trigonometric Series: A series of the form

$\frac{a_0}{2}$ + (a_1 cos x + b_1 sin x) + (a_2 cos 2x +b_2 sin 2x) + (a_3 cos 3x +b_3 sin 3x) +-------+ (a_n cos n x +b_n sin n x)

= $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n$ **cos nx + b_n sin nx) is called a trigonometric series** and the constants a_0 , a_n , b_n (n=0,1,2,3, ---------) are named its coefficients that are real and independent of x.

4.3 Fourier Series: A Fourier series is an exact expansion of periodic functions f(x) in terms of infinite sums of cosines and sines. This series utilises the orthogonality connection of the cosine and sine functions.

The Fourier transform can be called as generalization of the Fourier series. This term can also be applied to both the frequency domain representation and the mathematical function used. The Fourier transform supports in extending the Fourier series to non-periodic functions, which permits to view any function as a sum of simple sinusoids.

4.4 Definition: Let f(t) be a function defined for all real values of t. The **Fourier transform** of the function f(t) is represented by F{f(t)} and defined as $\int_{-\infty}^{\infty} f(t)e^{-ist}\,dt$, provided the integral exists and s is a parameter.

The Fourier integral is clearly a function of s and can be represented by $\bar{f}$(s) or simply by f(s)

We can say $\bar{f}$ (s) = F{f(t)} = $\int_{-\infty}^{\infty} f(t)e^{-ist}\,dt$

The **inverse formula** for infinite Fourier Transform is

f(t) = $F^{-1}\{\bar{f}$(s)} = $\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\, e^{ist}$ds

4.5 Fourier Sine and Cosine transform (or infinite Fourier Sine / cosine transform)

The infinite **Fourier Sine transform for the function** f(t), where 0 < t < ∞ denoted by F_s\{f(t)\} and is defined as

$\bar{f}_s(s) = F_s\{f(t)\} = \int_0^{\infty}$ f(t) sin st dt

The **inverse formula** for infinite Fourier Sine Transform is

f(t) = $Fs^{-1}\{\overline{fs}$(s)} = $\frac{2}{\pi}\int_0^{\infty} \bar{f}s$(s)sin st ds

and **Fourier Cosine transform** for the function f(t)

$\bar{f}_c(s) = F_c\{f(t)\} = \int_0^{\infty}$ f(t) cos st dt, s being a parameter.

The **inverse formula** for infinite Fourier cosine Transform is

f(t) = $Fc^{-1}\{\overline{fc}$(s)} = $\frac{2}{\pi}\int_0^{\infty} \bar{f}c$(s) cos st ds

4.6 Properties of Fourier transform : Some important properties of Fourier transform are as linear transform, modulation property, Parseval's theorem and duality.

Examples Based on Fourier Transform

Let us proceed further with some examples that involve integral and integral equation where Fourier transform is applied.

Example5.1 : Solve the Integral equation $\int_0^{\infty} f(x)\cos ⅄x$ =e$^{-⅄}$

Solution: It is given that $\int_0^{\infty} f(x)\cos ⅄x$ =e$^{-⅄}$

i.e. F_c\{f(x)\} = e$^{-⅄}$ => f(x) =F_c^{-1}\{ e$^{-⅄}$\}

i.e. f(x) = $\frac{2}{\pi}\int_0^{\infty}$ e$^{-⅄}$ cos ⅄x d⅄

[∵ $\int_0^{\infty}$ e^{-ax} cos bx dx = a /(a^2+b^2)]

= $\frac{2}{\pi}$. {1/(1+x^2) }

= 2 / π(1 + x^2)

Example 5.2 : Solve for f(x), the integral equation

$$\int_0^\infty f(x)\sin sx\,dx = \begin{cases} 1, & 0\le s<1 \\ 2, & 1\le s<2 \\ 0, & s\ge 2 \end{cases}$$

Solution: Here, it is given that

$$\int_0^\infty f(x)\sin sx\,dx = \begin{cases} 1, & 0\le s<1 \\ 2, & 1\le s<2 \\ 0, & s\ge 2 \end{cases}$$

We know that, $\bar{f}_s(s) = F_s\{f(x)\}$

$$= \int_0^\infty f(x)\sin sx\,dx = \begin{cases} 1, & 0\le s<1 \\ 2, & 1\le s<2 \\ 0, & s\ge 2 \end{cases}$$

∴ By inversion formula, $f(x) = \frac{2}{\pi}\int_0^\infty \overline{fs}(s)\sin sx\,ds$

$$= \frac{2}{\pi}\left[\int_0^1 \overline{fs}(s)\sin sx\,ds + \int_1^2 \overline{fs}(s)\sin sx\,ds + \int_2^\infty \overline{fs}(s)\sin sx\right]ds$$

$$= \frac{2}{\pi}\left[\int_0^1 \sin sx\,ds + \int_1^2 2\sin sx\,ds + 0\right]$$

$$= \frac{2}{\pi x}[1 + \cos x - 2\cos 2x]$$

Example 5. 3 : Using Parseval's identity, show that

$$\int_0^\infty \frac{dx}{(x^2+25)(x^2+81)} = \frac{\pi}{1260}$$

Solution:

Taking $f(x) = e^{-5x}$ and $g(x) = e^{-9x}$

∴ $F_c(s) = \frac{5}{(25+s^2)}$ and $G_c(s) = \frac{9}{(81+s^2)}$

Parseval's identity for Fourier Cosine transform is equal to

$$\frac{2}{\pi}\int_0^\infty F_c(s)\,G_c(s)\,ds = \int_0^\infty f(x)g(x)dx$$

Or $\frac{2}{\pi}\int_0^\infty \frac{45}{(s^2+25)(s^2+81)}\,ds = \int_0^\infty e^{-14x} = \frac{1}{14}$

Or $\frac{90}{\pi}\int_0^\infty \frac{1}{(s^2+25)(s^2+81)}\,ds = \frac{1}{14}$

Or $\int_0^\infty \frac{45}{(s^2+25)(s^2+81)}\,ds = \frac{\pi}{1260}$

Or $\int_0^\infty \frac{45}{(x^2+25)(x^2+81)}\,ds = \frac{\pi}{1260}$ [by changing the variables]

Hence the result

Example: Using Parseval's identity, show that

$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$$

Solution: Parseval's identity for Fourier Cosine transform

$= \frac{2}{\pi}\int_0^\infty [F_c(s)]^2\,ds = \int_0^\infty [f(x)]^2\,dx$ ----------------(1)

Taking $f(x) = e^{-x}$, x > 0

$F_c(s) = \int_0^\infty e^{-x}\cos sx\,ds = \frac{e^{-x}}{1+s^2}[-\cos sx + s\sin sx]_0^\infty$

$= 0 - \frac{-1}{1+s^2} = \frac{1}{1+s^2}$

From (1) ,we have

$\frac{2}{\pi}\int_0^\infty \{\frac{1}{1+s^2}\}^2\,ds = \int_0^\infty (e^{-x})^2\,dx = \frac{1}{2}$

Or $\int_0^\infty \frac{ds}{(1+s^2)^2} = \frac{\pi}{4}$

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Or $\int_0^\infty \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$ [Change of variable]

Now Let us study how to solve partial differential equation by using Fourier transform
Here we shall deal with the following types of partial differential equations:

Example 5.4: The temperature u in a semi-infinite rod is determined by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \; ; \quad 0 \le x < \infty$$

With the conditions:

(i) u =0 when t =0, x > 0

(ii) $\frac{\partial u}{\partial t}$ = - μ when x = 0

(iii) $\frac{\partial u}{\partial x} \to 0$ when $x \to \infty$

Determine the temperature formula.

Solution: Let u (x, t) denoted the temperature at any instant of time t.

Heat flow equation is $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$; $0 \le x < \infty$ -----------(1)

As $\frac{\partial u}{\partial x}$ at x=0 is given, so we use cosine transform to remove $\frac{\partial^2 u}{\partial x^2}$

Multiplying both sides of (1) by cos sx and integrating between the limits 0 to ∞,

$$\int_0^\infty \frac{\partial u}{\partial t} \cos sx \, dx = c^2 \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cos sx \, dx$$

We can write it as $\frac{\partial}{\partial t} \int_0^\infty u \cos sx \, dx = c^2 \left[\left(\cos sx \frac{\partial u}{\partial x}\right)_0^\infty - \int_0^\infty (-s \sin sx) \frac{\partial u}{\partial x} \, dx \right]$

(Integrating R.H.S.by parts)

$\frac{\partial u}{\partial x} \to 0$ when $x \to \infty$ (given)

$\therefore \frac{\partial}{\partial t} \int_0^\infty u \cos sx \, dx = c^2 \left[0 - \left(\frac{\partial u}{\partial x}\right)_{at\, x=0} + \int_0^\infty \frac{\partial u}{\partial x} \sin sx \, dx \right]$ ------------------(2)

As $\int_0^\infty u(x,t) \cos sx \, dx$ is the fourier cosine transform of u, we represent it by $\overline{u_c}$

Thus from (2) $\frac{d\overline{u_c}}{dt} = c^2 \left[\mu + s \int_0^\infty \frac{\partial u}{\partial x} \sin sx \, dx \right]$ $\left[\because \frac{\partial u}{\partial t} = -\mu \text{ when } x = 0\right]$

i.e. $\frac{d\overline{u_c}}{dt} = \mu c^2 + c^2 s \left[\{u \sin sx\}_0^\infty - \int_0^\infty u \, s \cos sx \, dx\right]$

or $\frac{d\overline{u_c}}{dt} = \mu c^2 - c^2 s^2 \left[0 - \int_0^\infty u \cos sx \, dx\right]$ [∵ $u \to 0$ as $x \to \infty$ by condition (i)]

or $\frac{d\overline{u_c}}{dt} = \mu c^2 - c^2 s^2 \overline{\mu_c}$ [Integral is Fourier cosine transform of u]

$\frac{d\overline{u_c}}{dt} + c^2 s^2 \overline{\mu_c} = \mu c^2$ --------------------------------(3)

Equation (3) is a linear differential equation of first order in $\overline{\mu_c}$

Integrating factors (I.F.) = $e^{\int c^2 s^2 \, dt} = e^{c^2 s^2 t}$

∴ The solution of (3) is $e^{c^2 s^2 t} \overline{u_c} = \int \mu c^2 e^{c^2 s^2 t} \, dt$ + A, where A is constant of integration

Or $e^{c^2 s^2 t} \overline{u_c} = \mu c^2 \frac{e^{c^2 s^2 t}}{c^2 s^2} + A$

$\therefore \overline{\mu_c}(s, t) = \frac{\mu}{s^2} + A e^{-s^2 c^2 t}$ ------------------------(4)

Now it is given that u(x,0) = 0 [∵ when t = 0, u = 0]

$\overline{\mu_c}(s,0) = 0$ ----------------------(5)

Taking t= 0 in (4) and putting $\overline{\mu_c}(s,0)$ in (5), we have

$\frac{\mu}{s^2} + A = 0$

$\therefore A = -\frac{\mu}{s^2}$

∴ from (4), we have $\overline{\mu_c}(s, t) = \frac{\mu}{s^2} - \frac{\mu}{s^2} e^{-s^2 c^2 t} = \frac{\mu}{s^2}(1 - e^{-s^2 c^2 t})$

Applying inverse Fourier cosine transform, we get

$$u(x, t) = \frac{2}{\pi} \int_0^\infty \frac{\mu}{s^2} [1 - e^{-c^2 s^2 t}] \cos sx \, dx$$

Thus we can produce more and more examples related to various areas of modern mathematics with application of FT.

Use of Fourier Transform in real life

The Fourier transform are not only supportive in the arena of Mathematics but supportive in real life problems also. Like, in audio processing, the Fourier Transform helps to identify the various frequencies present in an audio signal, supporting tasks like speech recognition, music classification, and noise reduction. In image analysis, the Fourier Transform can be applied to extract texture and pattern information from images. The Fourier Transform is an important image processing method used to break an image into its sine and cosine components.

The principle of Fourier Transform is applied in signal, such as the sound created by a musical instrument i.e. piano, violin, drum any sound recording can be exhibited as the sum of a collection of sine and cosine waves with different frequencies and amplitudes.

Fast Fourier Transform (FFT) algorithms expedite Fourier analysis Use of Fast Fourier Transform is seen widely in image processing and computer vision. For example, convolution, a fundamental image processing operation, may be performed speedily by usage of the Fast Fourier Transform. The Wiener filter, used to de-blur image, is expressed in terms of the Fourier transform.

Challenges and Futuristic Vision

Challenges may encompass computational complicacy and the requirement for accurate data sampling. Future progress may focus on enhancing FFT algorithms in emerging areas like quantum computing and advancing new mathematical technique to analyse non-linear systems.

Conclusion:

It is concluded that Fourier analysis is supportive in almost every aspect of the subject ranging from solution of linear differential equation to development of computer models, to the processing & analysis of data. The main benefit of Fourier analysis is that there is very little loss of information from the signal during the transformation. The Fourier transform maintains information on amplitude, harmonics, and phase and uses all parts of the waveform to transform the signal into the frequency domain. Fast Fourier transforms are applied to a great extent in engineering, music, science, and mathematics

Fourier Transform stands as a fundamental mathematical tool with profound applications across diverse fields. Its ability to decompose functions into frequency components revolutionizes mathematical analysis and drives innovations in science ,engineering and mathematics. Continued research and development in Fourier analysis promise further insights and advancements in understanding complex systems.

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