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THE UPPER TOTAL MONOPHONIC NUMBER OF A GRAPH

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ABSTRACT

A set *M* of vertices of a connected graph *G* is a monophonic set if every vertex of *G* lies on an *x*-y monophonic path for some elements *x* and *y* in *M.* The minimum cardinality of a monophonic set of *G* is the monophonic number of *G*,and is denoted by *m*(G). A monophonic set of cardinality *m*(G). is called a *m*-set of *G*. Any monophonic set of order *m*(G)is a minimum monophonic set of *G*. A monophonic set *M* in a connected graph *G* is called a minimal monophonic set if no proper subset of *M* is a monophonic set of *G*. The total monophonic set *M* of a graph *G* is a monophonic set *M* such that the subgraph induced by *M* has no isolated vertices, and is denoted by $m_t(G)$. The upper total monophonic set of a graph *G* is a minimal total monophonic set *M* such that the subgraph induced by *M* has no isolated vertices. The upper total monophonic number is the maximum cardinality of a minimal total monophonic set of *G*, and is denoted by $m_t^+(G)$. The upper monophonic numbers of some connected graphs are realized. It is proved that for any integers, *a*, *b* and *c* such that $2\leq a\leq b<c$, there exists a connected graph *G* with $m(G)=a$,$m_t(G)=b$ and $m_t^+(G)=c$.

Keywords: Monophonic set, monophonic number, total monophonic number, upper total monophonic number.

1. Introduction

By a graph *G* = (*V, E*) we mean a simple graph of order at least two. The order and size of *G* are denoted by *p* and *q*, respectively. For basic graph theoretic terminology, we refer to Harary [5]. The neighborhood of a vertex *v* is the set *N*(*v*) consisting of all vertices *u* which are adjacent with *v*. The closed neighborhood of a vertex *v* is the set *N*[*v*] = *N*(*v*) U{*v* }. A vertex *v* is an extreme vertex if the sub graph induced by its neighbors is complete. A vertex *v* is a semi-extreme vertex of *G* if the sub graph induced by its neighbors has a full degree vertex in *N*(*v*). In particular, every extreme vertex is a semi - extreme vertex and a semi - extreme vertex need not be an extreme vertex.

For any two vertices *x* and *y* in a connected graph *G*, the distance *d*(*x*, *y*) is the length of a shortest *x-y* path in *G*. An *x-y* path of length *d*(*x*, *y*) is called an *x-y* geodesic. A vertex *v* is said to lie on an *x-y* geodesic *P* if *v* is a vertex of *P* including the vertices *x* and *y*.

The closed interval $I[x, y]$ consists of all vertices lying on some x-y geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices is a geodetic set if $I[S]$ = V, and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. A geodetic set of cardinality $g(G)$ is called *a g*-set. The geodetic number of a graph was introduced in [1, 6] and further studied in [2, 3, 4, 5]. A set S of vertices of a graph G is an edge geodetic set if every edge of G lies on an x-y geodesic for some elements x and y in S. The minimum cardinality of an edge geodetic set of G is the edge geodetic number of G denoted by eg (G). The edge geodetic number was introduced and studied in [8]. The total edge of geodetic set of a graph G is an edge geodetic set S such that the subgraph induced by S has no isolated vertices. The minimum cardinality of a total edge geodetic set of G is the total edge geodetic number of G and is denoted by $eg_t(G)$.

A chord of a path $u_1, u_2, \ldots, u_k$ in G is an edge $u_i u_j$ with $j \geq i + 2$. A u-v path P is called a monophonic path of it is a chordless path. A set M of vertices is a monophonic set if every vertex of G lies on a monophonic path joining some pair of vertices in M, and the minimum cardinality of a monophonic set of G is the monophonic number of G, and is denoted by m(G). A monophonic set of cardinality m(G) is called a m-set of G. Any monophonic set of order m(G) is a minimum monophonic set of G. A monophonic set M in a connected graph G is called a minimal monophonic set if no proper subset of M is a monophonic set of G. The monophonic number of a graph G was studied in [9]. A monophonic set M in a connected graph G is called a minimal monophonic set if no proper subset of M is a monophonic set of G. The upper monophonic number $m^+(G)$ of G is the maximum cardinality of a minimal set of G. The upper monophonic number of a graph G was studied in [10]. The total monophonic set M of a graph G is a monophonic set M such that the subgraph induced by M has no isolated vertices, and is denoted by $m_t(G)$. The upper total monophonic set of a graph G is a minimal total monophonic set M such that the subgraph induced by M has no isolated vertices. The upper total monophonic number is the maximum cardinality of a minimal total monophonic set of G, and is denoted by $m_t^+(G)$.

The following Theorems will be used in the sequel.

Theorem 1.1 [9] : Each extreme vertex of a connected graph G belongs to every monophonic set of G.

Theorem 1.2 [10] : Let G be a connected graph with diameter d. Then $m(G) \leq p-d+1$. Throughout this paper G denotes a connected graph with atleast two vertices

2.UPPER TOTAL MONOPHOIC NUMBER OF A GRAPH

Definition 2.1: The total monoponic set M in a connected graph G is called a minimal total monophonic set of G if no proper subset of M is the total monophonic set of G. The upper total monophonic number $m_t^+(G)$ of G is the maximum cardinality of a minimal total monophonic set of G.

Example 2.2: For the graph G given in Figure2.*1*, $M_1 = \{v_2,v_4\}$, $M_2 = \{v_4,v_6\}$, $M_3 = \{v_2,v_5\}$ *are* the only three minimum monophonic sets of G , *so that* $m(G)=3$. *The set* $M_4 = \{v_1, v_3, v_5\}$ *and* $M_5=\{v_1,v_3,v_6\}$are minimalmonophonic sets of G. The set $M_6=\{v_1, v_3, v_4, v_5\}$, $M_7 = \{v_1,v_2,v_3,v_6\}$ are the minimal total monophonic sets of G, so that $m_t^+(G) \geq 4$. It is easily verified that no five elements of G is minimal total monophonic set of G and so $m_t^+(G) = 4$.

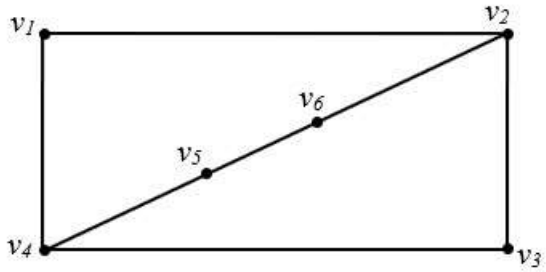

Figure 2.1 : G

Remark 2.3: Every minimum total monophonic set of G is a minimal total monophonic set of G and the converse is not true. For the graph G given in figure 2.1, $M_6 = \{v_1, v_3, v_4, v_5\}$ is a minimal total monophonic set of G but not a minimum total monophonic set of G.

Theorem 2.4 : For any connected graph G, $2 \leq m(G) \leq m^+{}_t(G) \leq p$.

Proof : Any monophonic set needs atleast 2 vertices and so $m(G) \geq 2$. Since every minimal total monophonic set is a monophonic set $m(G) \leq m_t^+(G)$. Also since $V(G)$ is a monophonic set of G, it is clear that $m_t^+(G) \leq p$. Thus $2 \leq m(G) \leq m_t^+(G) \leq p$.

Theorem 2.5 : For the complete graph K_P $(p \geq 2)$, $m_t^+(K_p) = m^+(K_p) = p$.

Proof : Since every vertx of the complete graph Kp $(p \geq 2)$ is an extreme vertex, the vertex set of Kp is the unique monoponic set. Thus $m^+(Kp) = m_t^+(Kp)$.

Theorem 2.6 : For a connected graph G of order p, the following are equivalent:

i. $m_t^+(G) = p$
ii. $m(G) = p$
iii. $G = K_p$

Proof : (i)=>(ii). Let $m^+{}_t(G) = p$. Then $M = V(G)$ is the unique minimal total monophonic set of G. Since no proper subset of M is a monophonic set, it is clear that M is the unique minimum total monopnic set of G and so $m(G) = p$.

(ii)=>(iii). Let $m(G) = p$. If $G \neq K_p$, then by theorem 1.3, $m(G) \leq p$-1, which is a condradiction. Therfore $G = K_{p.}$ (iii) $\Rightarrow$ (i). Let G = K_p. Then by Theorem 2.5, $m_t^+(G) = p$.

Theorem 2.6 : Let G be a connected graph with cut vertices and M be a minimal monoponic set of G. If v is a cut vertex of G, Then every component of G-v contains an element of M.

Proof : Suppose that there is a component G_1 of G-v such that G_1 contains no vertex of M. By Theorem 1.2, G_1 does not contain any end vertex of G. Thus G_1 contains atleast one vertex, say u. Since M is a minimal monophonic set , there exists vertices $x, y \in M$ such that u lies on the x-y monophonic path P : $x = u_0, u_1, u_2, \ldots, u, \ldots, u_t = y$ in G. Let P_1 be a x-u sub path of P and P_2 be a u-y subpath of P. Since v is a cut vertex of G, both P_1 and P_2 contain v so that P is not a path, which is a contradiction. Thus every component of G-v contains an element of M.

Theorem 2.7: For any connected graph G, no cut vertex of G belongs to any minimal total monophonic set of G.

Proof : Let M be a minimal total monophonic set of G and $v \in M$ be any vertex. We claim that v is not a cut vertex of G. Suppose that v is a cut vertex of G. Let $G_1, G_2, ..., G_r (r \geq 2)$ be the components of G-v. By theorem 2.6, each component $Gi, (1 \leq i \leq r)$ contains an element of M. We claim that $M_1 = M - \{v\}$ is also a monophonic set of G. Let x be a vertex of G. Since M is a minimal monophonic set of G, x lies on a monophonic path P joining a pair of vertices u and v of M. Assume without loss of generality that $u \in G_1$. Since v is adjacent to atleast one vertex of each $G_i (1 \leq i \leq r)$, assume that v is adjacent to z in G_k, $k \neq 1$. Since M is a monophonic set, z lies on a monophonic path Q joining v and a vertex w of M such that w must necessarily belongs to G_k. Thus $w \neq v$. Now, since v is a cut vertex of G, $P \cup Q$ is a path joining u and w in M and thus the vertex x lies on this monophonic path joining two vertices of M_1. Hence it follows that M_1 is a monophonic set of G. Since $M_1 \subsetneq M$, this contradicts the fact that M is a minimal total monophonic set of G. Hence $v \notin M$ so that no cut vertex of G belongs to any minimal total monophonic set of G.

Theorem 2.8 : For any Tree T with k end vertices, $m_t^+(T) = m^+(T) = m(T) = k$.

Proof: By Theorem 1.1, any monophonic set contains all the end vertices of T. By Theorem 2.7, no cut vertices of T belongs to a minimal total monophonic set of G. Hence it follows that, the set of all end vertices of T is the unique minimal total monophoni set of T so that $m^+(T) = m^+_t(T) = m(T) = k$.

Theorem 2.9: For a cycle $G = C_p (p \geq 4)$, $m^+_t(G) = 3$.

Proof : First suppose that $G = C_3$. It is a complete graph, by Theorem 2.5, we have $m^+_t(G) = 3$. For any cycle suppose that $m^+_t(G) > 3$, then there exist a minimal total monophonic set M_1 such that $|M_1| \geq 3$. Now it is clear that monophonic set $M \subsetneq M_1$, which is a contradiction to M_1 is a minimal total monophonic set of G. Therfore $m^+_t(G) = 3$.

Theorem 2.9: For the complete bipartite graph $G = K_{m,n}$.

(i) $m_t^+(G) = 2$ if $m = n = 1$

(ii) $m^+_t(G) = n+1$ if $m = 1, n \geq 2$

(iii) $m_t^+(G) = min\{m,n\}+1$, if $m,n \geq 2$.

Proof: (i) and (ii) follows from Theorem 2.7. (iii) Let $m,n \geq 2$. Assume without loss of generality that $m \leq n$. First assume that $m < n$. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, .., y_n\}$ be a bipartion of G. Let $M = Y$. We prove that M is a minimal total monophonic set of G. Any vertex y_i $(1 \leq i \leq n)$ lies on a monophonic path $y_i y_k$ for $k \neq 1$ so that M is a monophonic set of G. Let $M'' \subsetneq M \cdot$ Then there exists a vertex $y_j \in M$ such that $y_j \notin M'' \cdot$ Then the vertex y_j $(1 \leq j \leq m)$ does not lie on a monophonic path joining a pair of vertices of M'**.** Thus M' **is** not a**n** monophonic set of G. This shows that M is a minimal total monophonic set of G. Hence $m_t^+(G) \geq n$. Let M_1 be a minimal total monophonic set of G such that $|M_1| \geq n+1$. Since the vertex x_i $(1 \leq i \leq m$ and $1 \leq j \leq n)$ lies on a monophonic path $x_i x_k$ for any $k \neq i$, it follows that X is monophonic set of G. Hence M_1 cannot contain X. Similarly since Y is a minimal monophonic set of G, M1 cannot contain Y also. Hence $M_1 \subsetneq X' \cup Y'$ where $X^i \subsetneq X$ and $Y' \subsetneq Y$. Hence there exist a vertex $x_i \in X (1 \leq i \leq m)$ and a vertex $y_j \in Y$ $(1 \leq i \leq n)$ such that $x_i y_j \notin M_1$. Hence the edge $x_i y_j$ does not lie on a monophonic path joining a pair of vertices of M_1. It follows that M_1 is not a monophonic set of G, which is a contradiction. Thus M is a minimal total monophonic set of G. Hence $m_t^+(G) = \min\{m, n\}+1$.

3. Realization Results :

Theorem 3.1 : For positive integers r, d and $k \geq d + 2$ with $r \leq d \leq 2r$, there exists a connected graph G with rad $G = r$, diam $G = d$ and $m_t^+(G) = k$.

Proof : If $r = 1$, then $d = 1$ or 2. For $d = 1$, let $G = K_k$. Then by theorem 2.9 $m_t^+(G) = k$. Now, let $r \geq 2$. We construct a graph G with the desired properties as follows:

Case 1. $r = d$. Let $C_{2r}: u_1, u_2, \ldots u_{2r}, u_1$ be a cycle of order $2r$. Let G be the graph obtained by adding the new vertices $v_1, v_2, \ldots, v_{k-r-2}$ and joining each $v_i (1 \leq i \leq k-r-2)$ with u_1 and u_2 of c_{2r}. The graph G is as shown in Figure 3.1

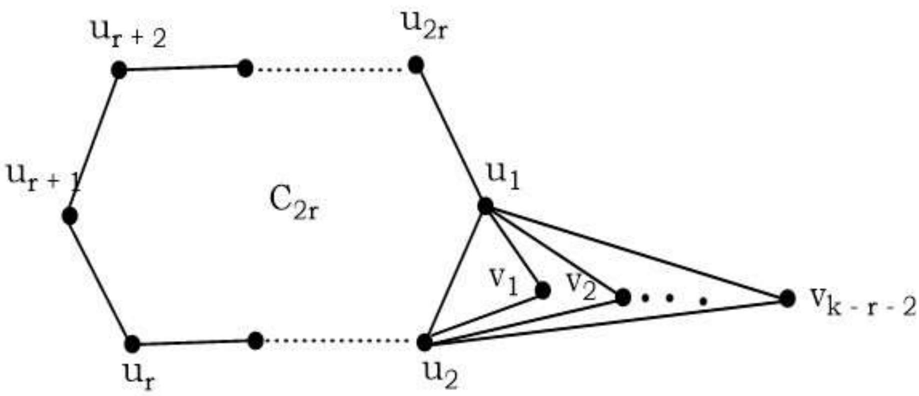

Figure 3.1 : G

It is easily verified that the eccentricity of each vertex of G is r so that rad G = diam G = r. If $k = r + 2$, then $G = c_{2r}$ and so by Theorem 2.7, $m_t^+(G) = r + 2 = k$.

If $k > r + 2$, then $M = \{v_1, v_2, \ldots, v_{k-r-2}\}$ is the set of all extreme vertices of G. It is clear that M is not a monophonic set of G. Let $M_1 = M \cup \{u_1, u_2, \ldots, u_r, u_{r+1}, u_{r+2})$. It is clear that M_1 is a minimum connected monophonic set of G ,also M_1 is also a minimal connected monophonic set of G so that $m^+(G) \geq |M_1| = k$.

It is clear that $M_1 = M \cup \{u_1, u_2, \ldots, u_r, u_{r+1}, u_{r+2})$ is a upper total monophonic set of G so that, $m_t^+(G) = k$.

Case 2 : r< d.

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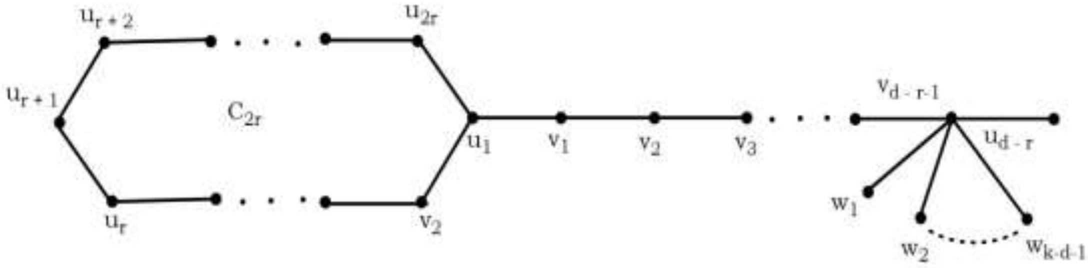

Figure 3.2 : G

Let $c_{2r} : u_1, u_2, \ldots, u_{2r}, u_1$ be a cycle of order $2r$ and let $P_{d-r+1} : v_0, v_1, \ldots, v_{d-r}$ be a path of order d-r+1. Let H be a graph obtained from c_{2r} and P_{d-r+1} by identifying u_1 in C_{2r} and v_0 in P_{d-r+1}. Now, we add k-d-1 new vertices $w_1, w_2, \ldots, w_{k-d-1}$ to the graph H and join each vertex w_i $(1 \leq i \leq k - d - 1)$ to the vertex v_{d-r-1} and also join u_r with u_{r+2} ,to obtain the graph G in Figure 3.2. Then rad $G = r$ and diam $G = d$.

Let $M' = \{v_0, v_1, \ldots, v_{d-r-1}, v_{d-r}, w_1, w_2, \ldots, w_{k-d-1}, u_{r+1}\}$. It is clear that M' is not a monophonic set of G. Let $M_1' = M' \cup \{u_2, u_3, \ldots, u_r\}$. It is clear that M_1' is a minimum connected monophonic set of G and so M_1' is also a minimal connected monophonic set of G. $m^+(G) = k$. It is clear that $M_1' = M'' \cup \{u_2, u_3, \ldots, u_r\}$ is a upper total monophonic set of G so that $m_t^+(G) = k$.

Theorem 3.2 : For any positive integers $2 \leq a \leq b < c$, there exists a connected graph G such that $m(G) = a$, $m^+(G) = b$, $m_t^+(G) = c$.

Proof : Take a copy of star $K_{1,a}$ with leaves $x_1, x_2, \ldots x_a$ and the support vertex x. Subdivide the edge xx_i, where $1 \leq i \leq c - b - 1$, calling the new vertices $y_1, y_2, \ldots, y_{c-b-1}$ where x_i is adjacent to y_i and y_i is adjacent to x for all $i \in \{1, 2, \ldots c\text{-}b\text{-} 1\}$. Let G be the graph obtained by adding $b - a$ new vertices $w_1, w_2, \ldots w_{b-a}$ and joining each w_i $(1 \leq i \leq b - a)$ with x, x_1. The graph G is shown in figure 3.3.

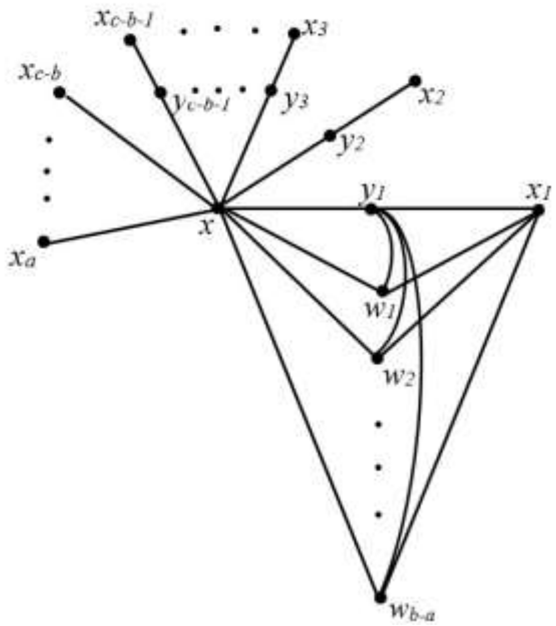

Figure 3.3 : G

Let first we show that $m(G) = a$. Let M be a monophonic set of G and let $W = \{x_2, x_3 \dots x_a\}$ be the set of all extreme vertices of G. It is clear that W is not a monophonic set of G. By theorem 1.1, every monophonic set of G contains W. Clearly $M = W \cup \{x_1\}$ is a monophonic set of G, so that $m(G) = a$.

Let $M_1 = M \cup \{w_1, w_2 \dots w_{b-a}\}$. Then M_1 is a minimal monophonic set of G. If M_1 is not a minimal monophonic set of G, then there is a proper subset T of M_1 such that T is a monophonic set of G. Then there exists $w \in M_1$ such that $w \notin T$. By theorem 1.1 $w \neq x_i$ $(1 \leq i \leq a)$. Therefore $w = w_i$ for some $i (1 \leq i \leq b - a)$. Since $w_i w_j$ $(1 \leq i, j \leq b - a)$, $i \neq j$ is a chord, w_i does not lie on a monophonic path joining some vertices of T and so T is not a monophonic set of G, which is a contradiction. Thus M_1 is a minimal monophonic set of G and so $m^+(G) = b$.

Let $M_2 = M_1 \cup \{y_1, y_2 \dots y_{c-b-1}, x\}$. It is clear that M_2 is a minimal total monophonic set of G, so that $m_t^+(G) = c$.

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