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ON THE COMPLETE PRODUCT OF HESITANCY FUZZY GRAPHS AND INTUITIONISTIC HESITANCY FUZZY GRAPHS

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Abstract
We present the concept of the complete product of a pair of hesitancy fuzzy graphs (HFG) and establish that the complete product of a pair of strong HFGs need not be a strong HFG. We define a new type of HFG namely the intuitionistic hesitancy fuzzy graph (IHFG) for which the complete product of a pair of strong IHFGs is a strong IHFG. For two complete IHFGs, their complete product is also a complete IHFG. Also we prove that if the complete product of a pair of IHFGs is strong, then at least one of the IHFG will be strong.

Keywords:
HFG, IHFG, strong IHFG, complete IHFG, complete product. **2020 Mathematics Subject Classification**: 05C72, 05C76

1. Introduction

Euler pioneered the notion of graph theory. A graph is an easy approach to express relationships between objects. Vertices of the graph represent objects, while edges describe relations. Designing a fuzzy network model is necessary when there is ambiguity in the description of the objects and their relations. Zadeh put forth the concept of fuzzy set [14]. Rosenfeld developed fuzzy graph (FG) theory in 1975 [10]. Atanassov [1] introduced intuitionistic fuzzy sets. R.Parvathi [7] developed intuitionistic fuzzy graph (IFG). T.Pathinathan [8] developed the concept of HFG. Many perspectives on hesitancy fuzzy sets and HFGs are discussed in [4,5,9,11,12,13]. Ch. Chaitanya and T.V. Pradeep Kumar [2] introduced the idea of complete product of FGs.

We define the complete product of a pair of HFGs. A HFG explains the degree of membership (MS), non-membership (NMS) and hesitancy of an element. In a HFG, the degree of hesitancy (ρ_1) of an element depends on the degree of MS (λ_1) and NMS (δ_1) of the element. A HFG is strong if it is λ-strong, δ-strong and ρ-strong. We establish that the complete product of a pair of strong HFGs need not be a strong HFG since the complete product of a pair of strong HFGs need not be ρ-strong. We introduce a new class of HFG, the intuitionistic HFG (IHFG) in which ρ_1 is independent of λ_1 and δ_1 and prove that the complete product of a pair of strong IHFGs is a strong IHFG. For two complete IHFGs, their complete product is also a complete IHFG. If the complete product of a pair of IHFGs is strong, then at least one of the IHFG will be strong. We refer to Harary [3] for fundamental graph theoretic terms.

2. Preliminaries

This section includes the basic definitions of graph, fuzzy graph, IFG and HFG.

Definition 2.1. [3] A graph $G = (V, E)$ consists of a vertex set V and an edge set E. Each edge has either one or two vertices connected to it, which are referred to as its end points.

Definition 2.2. [6] A fuzzy graph $G = (V, \sigma, \mu)$ where V is the vertex set, σ is a fuzzy subset of V and μ is a fuzzy relation on σ such that $\mu(u, v) \leq \sigma(u) \wedge \sigma(v),\ \forall\, u, v \in V$. V is finite and non-empty, μ is reflexive and symmetric.

Definition 2.3. [7] An IFG is $G = (V, E, \sigma, \mu)$, where V is the vertex set, $\sigma = (\lambda_1, \delta_1)$, $\mu = (\lambda_2, \delta_2)$ and $\lambda_1, \delta_1 : V \longrightarrow [0,1]$ represent the degree of MS, NMS of $v \in V$,

$$0 \leq \lambda_1(v) + \delta_1(v) \leq 1.$$

$\lambda_2, \delta_2 : V \times V \longrightarrow [0,1]$ represent the degree of MS, NMS of the edge $x = (u, v) \in V \times V$,

$$\lambda_2(x) \leq \lambda_1(u) \wedge \lambda_1(v) \qquad \delta_2(x) \leq \delta_1(u) \vee \delta_1(v)$$

$$0 \leq \lambda_2(x) + \delta_2(x) \leq 1, \forall\, x$$

Definition 2.4. [8] A HFG is $G = (V, E, \sigma, \mu)$, where V is the vertex set, $\sigma = (\lambda_1, \delta_1, \rho_1)$, $\mu = (\lambda_2, \delta_2, \rho_2)$ and $\lambda_1, \delta_1, \rho_1 : V \longrightarrow [0,1]$ represent the degree of MS, NMS and hesitancy of $v \in V$,

$$\lambda_1(v) + \delta_1(v) + \rho_1(v) = 1,$$

$$\text{where } \rho_1(v) = 1 - [\lambda_1(v) + \delta_1(v)]$$

$\lambda_2, \delta_2, \rho_2 : V \times V \longrightarrow [0,1]$ represent the degree of MS, NMS and hesitancy of $x \in V \times V$,

$$\lambda_2(x) \leq \lambda_1(u) \wedge \lambda_1(v)$$

$$\delta_2(x) \leq \delta_1(u) \vee \delta_1(v)$$

$$\rho_2(x) \leq \rho_1(u) \wedge \rho_1(v)$$

$$0 \leq \lambda_2(x) + \delta_2(x) + \rho_2(x) \leq 1, \forall\, x$$

3. Main Results

In HFG, the degree of hesitancy ρ_1 of an element v depends on the degree of MS (λ_1) and NMS (δ_1) of v. We define a new class of HFG namely, the intuitionistic HFG (IHFG) in which ρ_1 is independent of λ_1 and δ_1.

Definition 3.1. An IHFG is $G_\alpha = (V, E, \sigma, \mu)$, where V is the vertex set, $\sigma = (\lambda_1, \delta_1, \rho_1)$, $\mu = (\lambda_2, \delta_2, \rho_2)$ and $\lambda_1, \delta_1, \rho_1 : V \longrightarrow [0,1]$ represent the degree of MS, NMS and hesitancy of $v \in V$,

$$0 \leq \lambda_1(v) + \delta_1(v) + \rho_1(v) \leq 1$$

$\lambda_2, \delta_2, \rho_2 : V \times V \longrightarrow [0,1]$ represent the degree of MS, NMS and hesitancy of $x \in V \times V$, $\lambda_2(x) \leq \lambda_1(u) \wedge \lambda_1(v)$

$$\delta_2(x) \leq \delta_1(u) \vee \delta_1(v)$$

$$\rho_2(x) \leq \rho_1(u) \wedge \rho_1(v)$$

$$0 \leq \lambda_2(x) + \delta_2(x) + \rho_2(x) \leq 1, \forall\, x$$

Definition 3.2.[8] A HFG G or an IHFG G_α is

$$\lambda\text{-strong if } \lambda_2(x) = \lambda_1(u) \wedge \lambda_1(v)$$

$$\delta\text{-strong if } \delta_2(x) = \delta_1(u) \vee \delta_1(v)$$

$$\rho\text{-strong if } \rho_2(x) = \rho_1(u) \wedge \rho_1(v), \forall\, x \in E.$$

A HFG G or an IHFG G_α is strong if it is λ-strong, δ-strong and ρ-strong.

Definition 3.3. [8] A HFG G or an IHFG G_α is complete if

$\lambda_2(x) = \lambda_1(u) \wedge \lambda_1(v)$, $\delta_2(x) = \delta_1(u) \vee \delta_1(v)$, $\rho_2(x) = \rho_1(u) \wedge \rho_1(v)$, $\forall\, u, v \in V$.

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We can construct different types of products in HFGs and IHFGs as in fuzzy graphs, like tensor product, normal product, modular product, star product etc. But these products are defined on specific domains and not on the whole cartesian product $U \times V$ of the two vertex sets U and V of the two HFGs or IHFGs. Now we discuss the complete product of HFG and IHFG which is defined on the whole cartesian product.

Definition 3.4. The complete product of two HFGs, $G_1 = (U, E_U, \sigma, \mu)$, $G_2 = (V, E_V, \sigma', \mu')$ where $\sigma = (\lambda_1, \delta_1, \rho_1)$, $\mu = (\lambda_2, \delta_2, \rho_2)$, $\sigma' = (\lambda_1', \delta_1', \rho_1')$ and $\mu' = (\lambda_2', \delta_2', \rho_2')$ is the HFG $G = G_1 \circledast G_2 = (U \times V, E, \sigma \circledast \sigma', \mu \circledast \mu')$, $E = E_1 \cup E_2 \cup \ldots \ldots \ldots . . \cup E_8$ such that

$$E_1 = \{w : u_1 = u_2, w_2 \in E_V\}$$
$$E_2 = \{w : u_1 = u_2, w_2 \notin E_V\}$$
$$E_3 = \{w : v_1 = v_2, w_1 \in E_U\}$$
$$E_4 = \{w : v_1 = v_2, w_1 \notin E_U\}$$
$$E_5 = \{w : w_1 \in E_U, w_2 \notin E_V\}$$
$$E_6 = \{w : w_1 \notin E_U, w_2 \in E_V\}$$
$$E_7 = \{w : w_1 \in E_U, w_2 \in E_V\}$$

$E_8 = \{w : w_1 \notin E_U, w_2 \notin E_V\}$ where

$$w = \big((u_1, v_1), (u_2, v_2)\big), \quad w_1 = (u_1, u_2), w_2 = (v_1, v_2).$$
$$(\lambda_1 \circledast \lambda_1')\,(x) = \lambda_1(u) \wedge \lambda_1'(v)$$
$$(\delta_1 \circledast \delta_1')\,(x) = \delta_1(u) \vee \delta_1'(v)$$
$$(\rho_1 \circledast \rho_1')\,(x) = 1 - [\lambda_1(u) \wedge \lambda_1'(v) + \delta_1(u) \vee \delta_1'(v)], \text{ where } x = (u, v).$$

$$(\lambda_2 \circledast \lambda_2')(w) = \begin{cases} \lambda_1(u_1) \wedge \lambda_2'(w_2), & \text{if } w \in E_1 \\ \lambda_1(u_1) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2), & \text{if } w \in E_2 \\ \lambda_1'(v_1) \wedge \lambda_2(w_1), & \text{if } w \in E_3 \\ \lambda_1'(v_1) \wedge \lambda_1(u_1) \wedge \lambda_1(u_2), & \text{if } w \in E_4 \\ \lambda_2(w_1) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2), & \text{if } w \in E_5 \\ \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_2'(w_2), & \text{if } w \in E_6 \\ \lambda_2(w_1) \wedge \lambda_2'(w_2), & \text{if } w \in E_7 \\ \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2), & \text{if } w \in E_8 \end{cases} \qquad (1)$$

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$$(\delta_2 \circledast \delta_2')(w) = \begin{cases} \delta_1(u_1) \vee \delta_2'(w_2), & \text{if } w \in E_1 \\ \delta_1(u_1) \vee \delta_1'(v_1) \vee \delta_1'(v_2), & \text{if } w \in E_2 \\ \delta_1'(v_1) \vee \delta_2(w_1), & \text{if } w \in E_3 \\ \delta_1'(v_1) \vee \delta_1(u_1) \vee \delta_1(u_2), & \text{if } w \in E_4 \\ \delta_2(w_1) \vee \delta_1'(v_1) \vee \delta_1'(v_2), & \text{if } w \in E_5 \\ \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_2'(w_2), & \text{if } w \in E_6 \\ \delta_2(w_1) \vee \delta_2'(w_2), & \text{if } w \in E_7 \\ \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_1'(v_1) \vee \delta_1'(v_2), & \text{if } w \in E_8 \end{cases} \quad (2)$$

$$(\rho_2 \circledast \rho_2')(w) = \begin{cases} \rho_1(u_1) \wedge \rho_2'(w_2), & \text{if } w \in E_1 \\ \rho_1(u_1) \wedge \rho_1'(v_1) \wedge \rho_1'(v_2), & \text{if } w \in E_2 \\ \rho_1'(v_1) \wedge \rho_2(w_1), & \text{if } w \in E_3 \\ \rho_1'(v_1) \wedge \rho_1(u_1) \wedge \rho_1(u_2), & \text{if } w \in E_4 \\ \rho_2(w_1) \wedge \rho_1'(v_1) \wedge \rho_1'(v_2), & \text{if } w \in E_5 \\ \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho_2'(w_2), & \text{if } w \in E_6 \\ \rho_2(w_1) \wedge \rho_2'(w_2), & \text{if } w \in E_7 \\ \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho_1'(v_1) \wedge \rho_1'(v_2), & \text{if } w \in E_8 \end{cases} \quad (3)$$

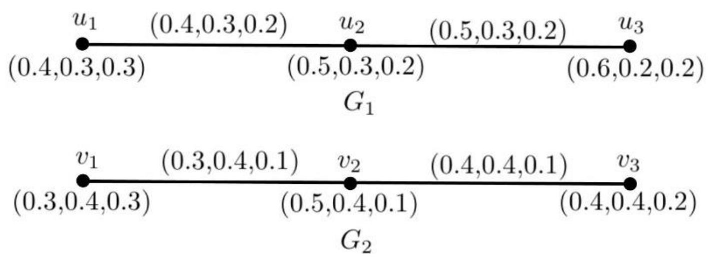

Figure 1. Strong HFGs G_1 and G_2

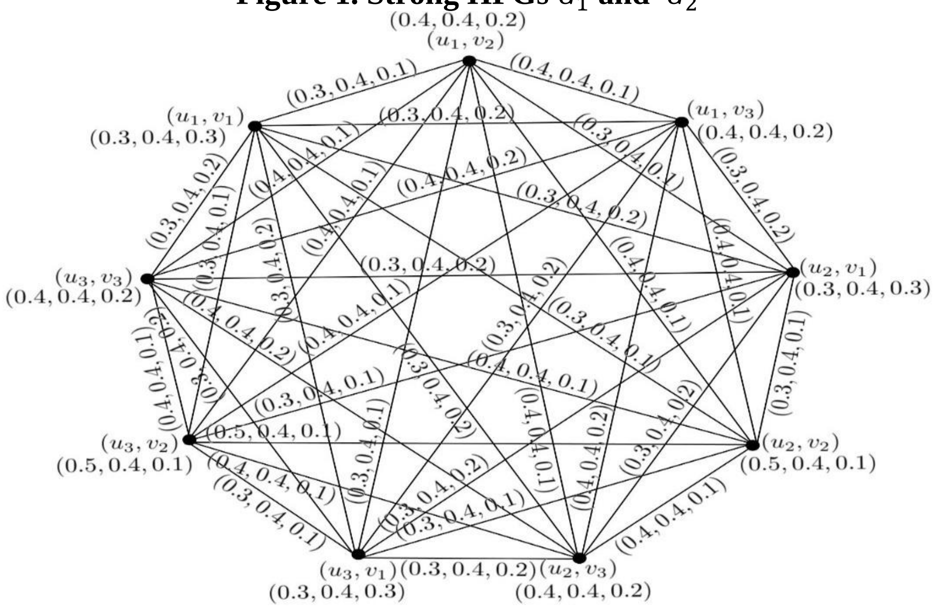

Figure 2. Complete Product of strong HFGs G_1 and G_2

Remark 3.5. For two strong HFGs G_1, G_2, their complete product $G_1 \circledast G_2$ need not be ρ-strong and hence need not be a strong HFG. In figure 1, G_1 and G_2 are two strong HFGs. Figure 2 is the complete product $G_1 \circledast G_2$.

$(\lambda_1 \circledast \lambda_1')\ (u_1, v_1) = \lambda_1(u_1) \wedge \lambda_1'(v_1) = 0.4 \wedge 0.3 = 0.3$

$(\delta_1 \circledast \delta_1')\ (u_1, v_1) = \delta_1(u_1) \vee \delta_1'(v_1) = 0.3 \vee 0.4 = 0.4$

$(\rho_1 \circledast \rho_1')\ (u_1, v_1) = 1 - (0.3 + 0.4) = 0.3$

$(\lambda_1 \circledast \lambda_1')\ (u_1, v_2) = \lambda_1(u_1) \wedge \lambda_1'(v_2) = 0.4 \wedge 0.5 = 0.4$

$(\delta_1 \circledast \delta_1')\ (u_1, v_2) = \delta_1(u_1) \vee \delta_1'(v_2) = 0.3 \vee 0.4 = 0.4$

$(\rho_1 \circledast \rho_1')\ (u_1, v_2) = 1 - (0.4 + 0.4) = 0.2$

Consider the edge $z = \big((u_1, v_1), (u_1, v_2)\big)$. Since $z \in E_1$,

$(\lambda_2 \circledast \lambda_2')\ (z) = \lambda_1(u_1) \wedge \lambda_2'(w_2) = 0.4 \wedge 0.3 = 0.3$

$(\delta_2 \circledast \delta_2')\ (z) = \delta_1(u_1) \vee \delta_2'(w_2) = 0.3 \vee 0.4 = 0.4$

$(\rho_2 \circledast \rho_2')\ (z) = \rho_1(u_1) \wedge \rho_2'(w_2) = 0.3 \wedge 0.1 = 0.1.$

i.e., for the edge z,

$(\lambda_2 \circledast \lambda_2')\ (z) = (\lambda_1 \circledast \lambda_1')\ (u_1, v_1) \wedge (\lambda_1 \circledast \lambda_1')\ (u_1, v_2)$. This is true for all the other edges in $G_1 \circledast G_2$ and hence $G_1 \circledast G_2$ is λ-strong.

$(\delta_2 \circledast \delta_2')\ (z) = (\delta_1 \circledast \delta_1')\ (u_1, v_1) \vee (\delta_1 \circledast \delta_1')\ (u_1, v_2)$, which is true for all the other edges and hence $G_1 \circledast G_2$ is δ-strong. $(\rho_2 \circledast \rho_2')\ (z) \neq (\rho_1 \circledast \rho_1')\ (u_1, v_1) \wedge (\rho_1 \circledast \rho_1')\ (u_1, v_2)$. i.e., $G_1 \circledast G_2$ is not ρ-strong and hence not a strong HFG.

Definition 3.6. Complete product of the IHFGs $G_{\alpha_1} = (U, E_U, \sigma, \mu)$, $G_{\alpha_2} = (V, E_V, \sigma', \mu')$ where $\sigma = (\lambda_1, \delta_1, \rho_1)$, $\mu = (\lambda_2, \delta_2, \rho_2)$, $\sigma' = (\lambda_1', \delta_1', \rho_1')$ and $\mu' = (\lambda_2', \delta_2', \rho_2')$ is the IHFG $G = G_{\alpha_1} \circledast G_{\alpha_2} = (U \times V, E, \sigma \circledast \sigma', \mu \circledast \mu')$, $E = E_1 \cup E_2 \cup \ldots \ldots \ldots . . \cup E_8$ with

$$(\lambda_1 \circledast \lambda_1')\ (x) = \lambda_1(u) \wedge \lambda_1'(v)$$
$$(\delta_1 \circledast \delta_1')\ (x) = \delta_1(u) \vee \delta_1'(v)$$
$$(\rho_1 \circledast \rho_1')\ (x) = \rho_1(u) \vee \rho_1'(v)$$

and the equations (1), (2) and (3).

Theorem 3.7. If G_{α_1} and G_{α_2} are two strong IHFGs, then their complete product $G_{\alpha_1} \circledast G_{\alpha_2}$ is also a strong IHFG.

Proof: Let $G_{\alpha_1}, G_{\alpha_2}$ be two strong IHFGs. Then, for $w_1 \in E_U$, $w_2 \in E_V$,

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$$\lambda_2(w_1) = \lambda_1(u_1) \wedge \lambda_1(u_2), \quad \lambda_2'(w_2) = \lambda_1'(v_1) \wedge \lambda_1'(v_2)$$
$$\delta_2(w_1) = \delta_1(u_1) \vee \delta_1(u_2), \quad \delta_2'(w_2) = \delta_1'(v_1) \vee \delta_1'(v_2)$$
$$\rho_2(w_1) = \rho_1(u_1) \wedge \rho_1(u_2), \quad \rho_2'(w_2) = \rho_1'(v_1) \wedge \rho_1'(v_2)$$

Case(i) When $w \in E_1$

$$(\lambda_2 \circledast \lambda_2')\,(w) = \lambda_1(u_1) \wedge \lambda_2'(w_2)$$
$$= \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2), \text{ since } \quad u_1 = u_2$$
$$= (\lambda_1 \circledast \lambda_1')\,(u_1, v_1) \wedge (\lambda_1 \circledast \lambda_1')\,(u_2, v_2)$$
$$(\delta_2 \circledast \delta_2')\,(w) = \delta_1(u_1) \vee \delta_2'(w_2)$$
$$= \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_1'(v_1) \vee \delta_1'(v_2), \text{ since } \quad u_1 = u_2$$
$$= (\delta_1 \circledast \delta_1')\,(u_1, v_1) \vee (\delta_1 \circledast \delta_1')\,(u_2, v_2)$$
$$(\rho_2 \circledast \rho_2')\,(w) = \rho_1(u_1) \wedge \rho_2'(w_2)$$
$$= \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho_1'(v_1) \wedge \rho_1'(v_2), \text{ since } \quad u_1 = u_2$$
$$= (\rho_1 \circledast \rho_1')\,(u_1, v_1) \wedge (\rho_1 \circledast \rho_1')\,(u_2, v_2)$$

Case(ii) When $w \in E_2$

$$(\lambda_2 \circledast \lambda_2')\,(w) = \lambda_1(u_1) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2)$$
$$= \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2), \text{ since } \quad u_1 = u_2$$
$$= (\lambda_1 \circledast \lambda_1')\,(u_1, v_1) \wedge (\lambda_1 \circledast \lambda_1')\,(u_2, v_2)$$
$$(\delta_2 \circledast \delta_2')\,(w) = \delta_1(u_1) \vee \delta_1'(v_1) \vee \delta_1'(v_2)$$
$$= \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_1'(v_1) \vee \delta_1'(v_2), \text{ since } \quad u_1 = u_2$$
$$= (\delta_1 \circledast \delta_1')\,(u_1, v_1) \vee (\delta_1 \circledast \delta_1')\,(u_2, v_2)$$
$$(\rho_2 \circledast \rho_2')\,(w) = \rho_1(u_1) \wedge \rho_1'(v_1) \wedge \rho_1'(v_2)$$
$$= \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho_1'(v_1) \wedge \rho_1'(v_2), \text{ since } \quad u_1 = u_2$$
$$= (\rho_1 \circledast \rho_1')\,(u_1, v_1) \wedge (\rho_1 \circledast \rho_1')\,(u_2, v_2)$$

Case(iii) When $w \in E_3$

$$(\lambda_2 \circledast \lambda_2')\,(w) = \lambda_1'(v_1) \wedge \lambda_2(w_1)$$
$$= \lambda_1'(v_1) \wedge \lambda_1'(v_2) \wedge \lambda_1(u_1) \wedge \lambda_1(u_2), \text{ since } \quad v_1 = v_2$$
$$= (\lambda_1 \circledast \lambda_1')\,(u_1, v_1) \wedge (\lambda_1 \circledast \lambda_1')\,(u_2, v_2)$$
$$(\delta_2 \circledast \delta_2')\,(w) = \delta_1'(v_1) \vee \delta_2(w_1)$$
$$= \delta_1'(v_1) \vee \delta_1'(v_2) \vee \delta_1(u_1) \vee \delta_1(u_2), \text{ since } \quad v_1 = v_2$$
$$= (\delta_1 \circledast \delta_1')\,(u_1, v_1) \vee (\delta_1 \circledast \delta_1')\,(u_2, v_2)$$

$$(\rho_2 \circledast \rho_2')\,(w) = \rho_1'(v_1) \wedge \rho_2(w_1)$$
$$= \rho_1'(v_1) \wedge \rho_1'(v_2) \wedge \rho_1(u_1) \wedge \rho_1(u_2), \text{ since } \; v_1 = v_2$$
$$= (\rho_1 \circledast \rho_1')\,(u_1, v_1) \wedge (\rho_1 \circledast \rho_1')\,(u_2, v_2)$$

Case(iv) When $w \in E_4$

$$(\lambda_2 \circledast \lambda_2')\,(w) = \lambda_1'(v_1) \wedge \lambda_1(u_1) \wedge \lambda_1(u_2)$$
$$= \lambda_1'(v_1) \wedge \lambda_1'(v_2) \wedge \lambda_1(u_1) \wedge \lambda_1(u_2), \text{ since } \; v_1 = v_2$$
$$= (\lambda_1 \circledast \lambda_1')\,(u_1, v_1) \wedge (\lambda_1 \circledast \lambda_1')\,(u_2, v_2)$$
$$(\delta_2 \circledast \delta_2')\,(w) = \delta_1'(v_1) \vee \delta_1(u_1) \vee \delta_1(u_2)$$
$$= \delta_1'(v_1) \vee \delta_1'(v_2) \vee \delta_1(u_1) \vee \delta_1(u_2), \text{ since } \; v_1 = v_2$$
$$= (\delta_1 \circledast \delta_1')\,(u_1, v_1) \vee (\delta_1 \circledast \delta_1')\,(u_2, v_2)$$
$$(\rho_2 \circledast \rho_2')\,(w) = \rho_1'(v_1) \wedge \rho_1(u_1) \wedge \rho_1(u_2)$$
$$= \rho_1'(v_1) \wedge \rho_1'(v_2) \wedge \rho_1(u_1) \wedge \rho_1(u_2), \text{ since } \; v_1 = v_2$$
$$= (\rho_1 \circledast \rho_1')\,(u_1, v_1) \wedge (\rho_1 \circledast \rho_1')\,(u_2, v_2)$$

Case(v) When $w \in E_5$

$$(\lambda_2 \circledast \lambda_2')\,(w) = \lambda_2(w_1) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2)$$
$$= \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2)$$
$$= (\lambda_1 \circledast \lambda_1')\,(u_1, v_1) \wedge (\lambda_1 \circledast \lambda_1')\,(u_2, v_2)$$
$$(\delta_2 \circledast \delta_2')\,(w) = \delta_2(w_1) \vee \delta_1'(v_1) \vee \delta_1'(v_2)$$
$$= \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_1'(v_1) \vee \delta_1'(v_2)$$
$$= (\delta_1 \circledast \delta_1')\,(u_1, v_1) \vee (\delta_1 \circledast \delta_1')\,(u_2, v_2)$$
$$(\rho_2 \circledast \rho_2')\,(w) = \rho_2(w_1) \wedge \rho_1'(v_1) \wedge \rho_1'(v_2)$$
$$= \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho_1'(v_1) \wedge \rho_1'(v_2)$$
$$= (\rho_1 \circledast \rho_1')\,(u_1, v_1) \wedge (\rho_1 \circledast \rho_1')\,(u_2, v_2)$$

Case(vi) When $w \in E_6$

$$(\lambda_2 \circledast \lambda_2')\,(w) = \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_2'(w_2)$$
$$= \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2)$$
$$= (\lambda_1 \circledast \lambda_1')\,(u_1, v_1) \wedge (\lambda_1 \circledast \lambda_1')\,(u_2, v_2)$$
$$(\delta_2 \circledast \delta_2')\,(w) = \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_2'(w_2)$$
$$= \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_1'(v_1) \vee \delta_1'(v_2)$$
$$= (\delta_1 \circledast \delta_1')\,(u_1, v_1) \vee (\delta_1 \circledast \delta_1')\,(u_2, v_2)$$

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$$(\rho_2 \circledast \rho_2')\,(w) = \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho_2'(w_2)$$
$$= \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho_1'(v_1) \wedge \rho_1'(v_2)$$
$$= (\rho_1 \circledast \rho_1')\,(u_1, v_1) \wedge (\rho_1 \circledast \rho_1')\,(u_2, v_2)$$

Case(vii) When $w \in E_7$

$$(\lambda_2 \circledast \lambda_2')\,(w) = \lambda_2(w_1) \wedge \lambda_2'(w_2)$$
$$= \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2)$$
$$= (\lambda_1 \circledast \lambda_1')\,(u_1, v_1) \wedge (\lambda_1 \circledast \lambda_1')\,(u_2, v_2)$$
$$(\delta_2 \circledast \delta_2')\,(w) = \delta_2(w_1) \vee \delta_2'(w_2)$$
$$= \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_1'(v_1) \vee \delta_1'(v_2)$$
$$= (\delta_1 \circledast \delta_1')\,(u_1, v_1) \vee (\delta_1 \circledast \delta_1')\,(u_2, v_2)$$
$$(\rho_2 \circledast \rho_2')\,(w) = \rho_2(w_1) \wedge \rho_2'(w_2)$$
$$= \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho_1'(v_1) \wedge \rho_1'(v_2)$$
$$= (\rho_1 \circledast \rho_1')\,(u_1, v_1) \wedge (\rho_1 \circledast \rho_1')\,(u_2, v_2)$$

Case(viii) When $w \in E_8$

$$(\lambda_2 \circledast \lambda_2')\,(w) = \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2)$$
$$= (\lambda_1 \circledast \lambda_1')\,(u_1, v_1) \wedge (\lambda_1 \circledast \lambda_1')\,(u_2, v_2)$$
$$(\delta_2 \circledast \delta_2')\,(w) = \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_1'(v_1) \vee \delta_1'(v_2)$$
$$= (\delta_1 \circledast \delta_1')\,(u_1, v_1) \vee (\delta_1 \circledast \delta_1')\,(u_2, v_2)$$
$$(\rho_2 \circledast \rho_2')\,(w) = \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho_1'(v_1) \wedge \rho_1'(v_2)$$
$$= (\rho_1 \circledast \rho_1')\,(u_1, v_1) \wedge (\rho_1 \circledast \rho_1')\,(u_2, v_2)$$

Thus, $G = G_{\alpha_1} \circledast G_{\alpha_2}$ is a strong IHFG.

Example 3.8. In figure 3, G_{α_1} and G_{α_2} are two strong IHFGs. Their complete product $G_{\alpha_1} \circledast G_{\alpha_2}$ shown in figure 4 is a strong IHFG since all the edges are strong edges.

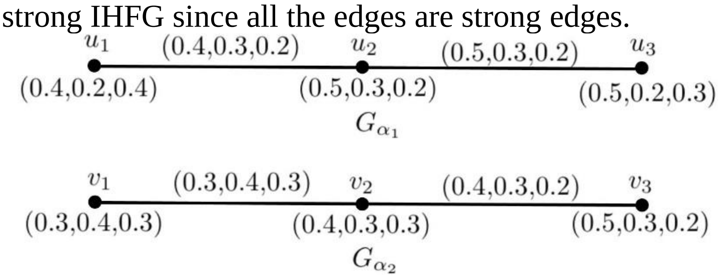

Figure 3. Strong IHFGs G_{α_1} and G_{α_2}

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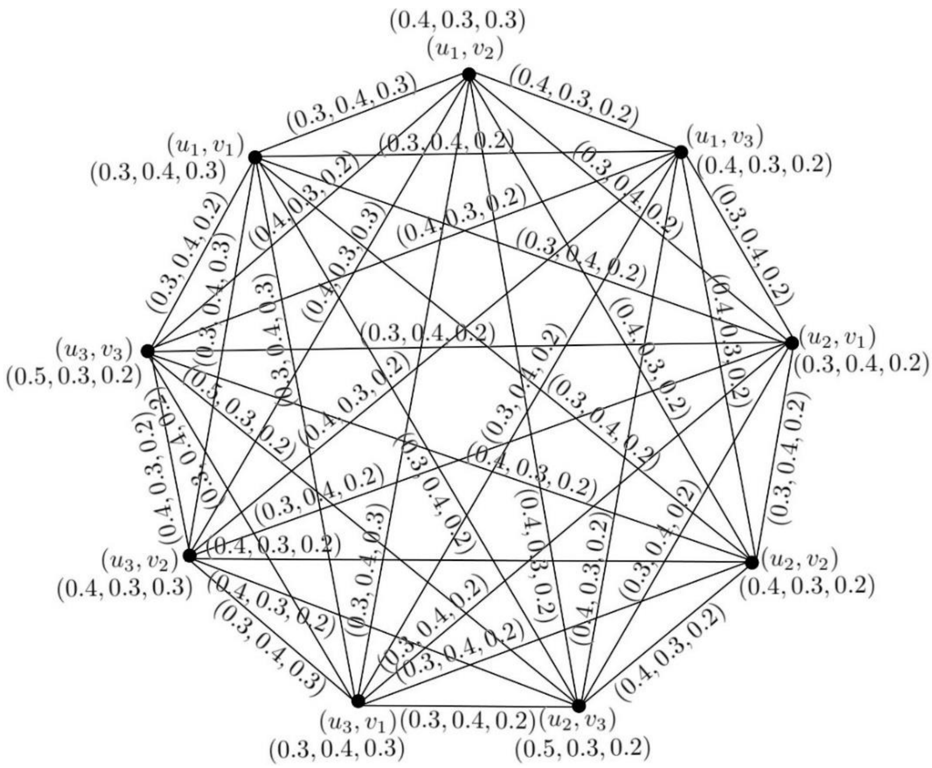

Figure 4. Complete Product of strong IHFGs G_{α_1} and G_{α_2}

Theorem 3.9. If G_{α_1} and G_{α_2} are two complete IHFGs, then their complete product $G_{\alpha_1} \circledast G_{\alpha_2}$ is also a complete IHFG.

Proof: Similar to 3.7

Theorem 3.10. If $G_{\alpha_1}, G_{\alpha_2}$ are two IHFGs such that $G_{\alpha_1} \circledast G_{\alpha_2}$ is strong, then at least one of G_{α_1} or G_{α_2} will be strong.

Proof: Assume that the two IHFGs $G_{\alpha_1}, G_{\alpha_2}$ are not strong. Then there exists at least one $w_1 \in E_U$, $w_2 \in E_V$ with

$$\lambda_2(w_1) < \lambda_1(u_1) \wedge \lambda_1(u_2), \quad \lambda_2'(w_2) < \lambda_1'(v_1) \wedge \lambda_1'(v_2)$$
$$\delta_2(w_1) < \delta_1(u_1) \vee \delta_1(u_2), \quad \delta_2'(w_2) < \delta_1'(v_1) \vee \delta_1'(v_2)$$
$$\rho_2(w_1) < \rho_1(u_1) \wedge \rho_1(u_2), \quad \rho_2'(w_2) < \rho_1'(v_1) \wedge \rho_1'(v_2)$$

When $w \in E_1$. Then,

$$(\lambda_2 \circledast \lambda_2')\,(w) = \lambda_1(u_1) \wedge \lambda_2'(w_2)$$
$$< \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2), \text{ since } \quad u_1 = u_2$$
$$= (\lambda_1 \circledast \lambda_1')\,(u_1, v_1) \wedge (\lambda_1 \circledast \lambda_1')\,(u_2, v_2)$$

i.e., $(\lambda_2 \circledast \lambda_2')\,(w) < (\lambda_1 \circledast \lambda_1')\,(u_1, v_1) \wedge (\lambda_1 \circledast \lambda_1')\,(u_2, v_2)$

$(\delta_2 \circledast \delta_2')\ (w) = \delta_1(u_1) \vee \delta_2'(w_2)$

$< \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_1'(v_1) \vee \delta_1'(v_2)$, since $u_1 = u_2$

$= (\delta_1 \circledast \delta_1')\ (u_1, v_1) \vee (\delta_1 \circledast \delta_1')\ (u_2, v_2)$

i.e., $(\delta_2 \circledast \delta_2')\ (w) < (\delta_1 \circledast \delta_1')\ (u_1, v_1) \vee (\delta_1 \circledast \delta_1')\ (u_2, v_2)$

$(\rho_2 \circledast \rho_2')\ (w) = \rho_1(u_1) \wedge \rho_2'(w_2)$

$< \rho_1(u_1) \wedge \rho_1(u_2) \wedge \rho_1'(v_1) \wedge \rho_1'(v_2)$, since $u_1 = u_2$

$= (\rho_1 \circledast \rho_1')\ (u_1, v_1) \wedge (\rho_1 \circledast \rho_1')\ (u_2, v_2)$

i.e., $(\rho_2 \circledast \rho_2')\ (w) < (\rho_1 \circledast \rho_1')\ (u_1, v_1) \wedge (\rho_1 \circledast \rho_1')\ (u_2, v_2)$

Hence $G_{\alpha_1} \circledast G_{\alpha_2}$ is not strong, a contradiction. So at least one of G_{α_1} or G_{α_2} will be strong.

4. Application

IHFGs can be suitably used in real life problems. It can work as a good aid in solving companies' merger problems. Consider two strong networks G_{α_1} and G_{α_2} in figure 3, with vertices representing different companies. The membership degree of the vertices indicates the market worth of the companies and the membership degree of the edges indicates the market worth of the companies' joint ventures. Since G_{α_1} and G_{α_2} are strong, all the edges in G_{α_1} and G_{α_2} are strong and all the edges in the complete product $G_{\alpha_1} \circledast G_{\alpha_2}$ are also strong. As the complete product is defined on the whole cartesian product, it includes all the possible edges between every pair of vertices. Thus, this product is stronger and more reliable than other products and the decision on merger problems based on this result will be more accurate.

5. Conclusion

IHFGs offers a wide range of uses in the fields of robotics, artificial intelligence and medical diagnosis. We investigated a novel product known as the complete product of two IHFGs, which accounts for all potential edges. We proved that the complete product of two strong IHFGs is a strong IHFG and the complete product of two complete IHFGs is a complete IHFG. Also, we proved that if the complete product of a pair of IHFGs is strong, then at least one of the IHFG will be strong. IHFG models provide exact and accurate outcomes for making decisions and resolving merger related problems. Our future work is to broaden the scope of our investigation to study the complement of the complete product of IHFG.

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