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STUDY OF NEW CLASS USING SALAGEAN DIFFERENTIAL OPERATOR WITH SOME MISSING COEFFICIENTS

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Abstract
In this paper, we study a new class $\varphi_n(\alpha,\beta,\gamma,\lambda)$ using the Salagean differential operator. Also obtained various properties of given class such as coefficient estimates, convex linear combination, extreme points and their related results.

Keywords*:* Analytic functions, Univalent functions, Salagean derivative
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Introduction

Let A_{uf} denote the class of analytic univalent functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_{2k} z^{2k} \tag{1}$$

defined in the unit disc $U = \{z : |z| < 1\}$.

Let φ_n denote the subclass of A_{uf} in U, consisting of analytic functions whose non-zero coefficients from the second onwards are negative. That is, an analytic function $f \in \varphi_n$, if it has a Taylor expansion of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_{2k} z^{2k}, \qquad (a_{2k} \geq 0) \tag{2}$$

which are univalent in the open disc U.

For $f \in A_{uf}$, Salagean [1], introduced the following operator D^n which is called the Salagean differential operator.

$$D^0 f(z) = f(z), \quad D' f(z) = f'(z)$$

$$D^n f(z) = D(D^{n-1} f(z)), \qquad n \in N.$$

We note that,

$$D^n f(z) = z + \sum_{k=2}^{\infty} (2k)^n a_{2k} z^{2k}, \qquad n \in N_0. \tag{3}$$

Now by Salagean differential operator, we describe the following subclass of φ_n.

Let $\varphi_n(\alpha,\beta,\gamma,\lambda)$ be the subclass of φ_n consisting of functions which satisfy the conditions

$$R\left\{\frac{1 - z(D^n f)'}{1 - z(D^n f)' + (1-\lambda)(D^n f) - (\alpha - \beta)}\right\} > \gamma \tag{4}$$

for some $0 \leq \alpha < 1, 0 \leq \beta < 1, 0 \leq \gamma < 1, 0 \leq \lambda < 1$ and $n \in N_0$.

Coefficient Estimates

Theorem 2.1: A function f defined by (2) is in the class $\varphi_n(\alpha,\beta,\gamma,\lambda)$ if and only if

$$\sum_{k=2}^{\infty}(2k)^n a_{2k}[2k\gamma-2k+\gamma\lambda-\gamma]<\gamma(\alpha-\beta+\lambda-1) \qquad (2.1)$$

Where $0\le\alpha<1, 0\le\beta<1, 0\le\gamma<1, 0\le\lambda<1$ and $n\in N_0$.

Proof: Suppose $f\in\varphi_n(\alpha,\beta,\gamma,\lambda)$. Then

$$R\left\{\frac{1-z(D^n f)'}{1-z(D^n f)'+(1-\lambda)(D^n f)-(\alpha-\beta)}\right\}>\gamma$$

$$R\left\{\frac{1-z+\sum_{k=2}^{\infty}(2k)^n 2ka_{2k}z^{2k}}{1-z+\sum_{k=2}^{\infty}(2k)^n 2ka_{2k}z^{2k}+(1-\lambda)(z-\sum_{k=2}^{\infty}(2k)^n 2ka_{2k}z^{2k})-(\alpha-\beta)}\right\}>\gamma .$$

Letting $z\to 1$, then we get

$$\left\{\frac{\sum_{k=2}^{\infty}(2k)^n 2ka_{2k}}{\sum_{k=2}^{\infty}(2k)^n 2ka_{2k}+(1-\lambda)(1-\sum_{k=2}^{\infty}(2k)^n a_{2k})-(\alpha-\beta)}\right\}>\gamma .$$

$$\sum_{k=2}^{\infty}(2k)^n 2ka_{2k}>\gamma\sum_{k=2}^{\infty}(2k)^n 2ka_{2k}+\gamma-\gamma\sum_{k=2}^{\infty}(2k)^n a_{2k}-\gamma\lambda+\gamma\lambda\sum_{k=2}^{\infty}(2k)^n a_{2k}-\gamma(\alpha-\beta)$$

$$\sum_{k=2}^{\infty}(2k)^n a_{2k}[\gamma 2k-\gamma+\gamma\lambda-2k]<\gamma(\alpha-\beta)-\gamma+\gamma\lambda$$

$$\sum_{k=2}^{\infty}(2k)^n a_{2k}[2k\gamma-2k+\gamma\lambda-\gamma]<\gamma(\alpha-\beta+\lambda-1) .$$

Conversely assume that (2.1) be true, we have to show that (4) is satisfied or equivalently,

$$\left|\left\{\frac{1-z(D^n f)'}{1-z(D^n f)'+(1-\lambda)(D^n f)-(\alpha-\beta)}\right\}-1\right|<1-\gamma .$$

But

$$\left|\left\{\frac{1-z+\sum_{k=2}^{\infty}(2k)^n 2ka_{2k}z^{2k}}{1-z+\sum_{k=2}^{\infty}(2k)^n 2ka_{2k}z^{2k}+(1-\lambda)(z-\sum_{k=2}^{\infty}(2k)^n 2ka_{2k}z^{2k})-(\alpha-\beta)}\right\}-1\right|$$

$$\le\frac{\sum_{k=2}^{\infty}(2k)^n a_{2k}(1-\lambda)-(\lambda-1)-(\alpha-\beta)}{\sum_{k=2}^{\infty}(2k)^n a_{2k}(2k-1+\lambda)+(1-\lambda-(\alpha-\beta))} .$$

The last expression is bounded above by $1-\gamma$ if

$$\sum_{k=2}^{\infty}(2k)^n a_{2k}(1-\lambda)-(\lambda-1)-(\alpha-\beta)$$
$$\leq(1-\gamma)[\sum_{k=2}^{\infty}(2k)^n a_{2k}(2k-1+\lambda)+(1-\lambda-(\alpha-\beta))]$$

Or

$$\sum_{k=2}^{\infty}(2k)^n a_{2k}[2k\gamma-2k+\gamma\lambda-\gamma]<\gamma(\alpha-\beta+\lambda-1)$$

which is true by hypothesis. This completes the assertion of Theorem 2.1.

Corollary 2.2: If $f\in\varphi_n(\alpha,\beta,\gamma,\lambda)$ then

$$|a_{2k}|\leq\frac{\gamma(\alpha-\beta+\lambda-1)}{(2k)^n[2k\gamma-2k+\gamma\lambda-\gamma]}.$$

Proof: From the Theorem 2.1,

$$\sum_{k=2}^{\infty}(2k)^n[2k\gamma-2k+\gamma\lambda_2-\gamma]a_{2k}$$
$$\leq\sum_{k=2}^{\infty}(2k)^n[2k\gamma-2k+\gamma\lambda_1-\gamma]a_{2k}$$
$$\leq\gamma(\alpha-\beta+\lambda-1)$$

For $f(z)\in\varphi_n(\alpha,\beta,\gamma,\lambda_2)$.

Hence $f(z)\in\varphi_n(\alpha,\beta,\gamma,\lambda_1)$.

Theorem 2.3: Let $f(z)\in\varphi_n(\alpha,\beta,\gamma,\lambda)$. Define $f_1(z)=z$ and

$$f_{2k}(z)=z+\frac{\gamma(\alpha-\beta+\lambda-1)}{(2k)^n[2k\gamma-2k+\gamma\lambda-\gamma]}z^{2k},\quad k=2,3,\ldots$$

for some $0\leq\alpha<1, 0\leq\beta<1, 0\leq\gamma<1, 0\leq\lambda<1$ and $n\in N_0$ and $z\in U$, $f\in\varphi_n(\alpha,\beta,\gamma,\lambda)$

if and only if f can be expressed as $f(z)=\sum_{k=1}^{\infty}u_{2k}f_{2k}(z)$ where $u_{2k}\geq 0$ and $\sum_{k=1}^{\infty}u_{2k}=1$.

Proof: If $f(z)=\sum_{k=1}^{\infty}u_{2k}f_{2k}(z)$ with $\sum_{k=1}^{\infty}u_{2k}=1, u_{2k}\geq 0$, then

$$\sum_{k=2}^{\infty}\frac{(2k)^n[2k\gamma-2k+\gamma\lambda-\gamma]u_{2k}}{(2k)^n[2k\gamma-2k+\gamma\lambda-\gamma]}(\gamma[\alpha-\beta+\lambda-1])$$
$$=\sum_{k=2}^{\infty}u_{2k}\gamma(\alpha-\beta+\lambda-1)$$
$$=(1-u_2)\gamma(\alpha-\beta+\lambda-1)$$
$$\leq\gamma(\alpha-\beta+\lambda-1).$$

Hence $f\in\varphi_n(\alpha,\beta,\gamma,\lambda)$. Conversely, let

$$f(z)=z-\sum_{k=2}^{\infty}a_{2k}z^{2k}\in\varphi_n(\alpha,\beta,\gamma,\lambda)$$

define

$$u_{2k}=\frac{(2k)^n[2k\gamma-2k+\gamma\lambda-\gamma]|a_{2k}|}{\gamma(\alpha-\beta+\lambda-1)},\quad k=2,3,\ldots$$

and define $u_2 = 1 - \sum_{k=2}^{\infty} u_{2k}$.

From Theorem 2.1, $\sum_{k=1}^{\infty} u_{2k} \le 1$ and so $u_2 \ge 0$. Since

$$u_{2k} f_{2k}(z) = u_{2k} f + a_{2k} z^{2k}$$

$$\sum_{k=1}^{\infty} u_{2k} f_{2k}(z) = z - \sum_{k=2}^{\infty} a_{2k} z^{2k} = f(z).$$

Theorem 2.4: The class $\varphi_n(\alpha,\beta,\gamma,\lambda)$ is closed under convex linear combination.

Proof: Let $f, g \in S_n(\alpha,\beta,\gamma,\lambda)$ and let

$$f(z) = z - \sum_{k=2}^{\infty} a_{2k} z^{2k}$$

and $$g(z) = z - \sum_{k=2}^{\infty} b_{2k} z^{2k}.$$

For ξ such that $0 \le \xi \le 1$, it suffices to show that the function define by

$$h(z) = (1-\xi) f(z) + \xi g(z),$$

$z \in U$ belongs to $\varphi_n(\alpha,\beta,\gamma,\lambda)$. Now,

$$h(z) = z - \sum_{k=2}^{\infty} [(1-\xi) a_{2k} + \xi b_{2k}] z^{2k}.$$

Apply Theorem 2.1, for $f, g \in \varphi_n(\alpha,\beta,\gamma,\lambda)$. We have,

$$\sum_{k=2}^{\infty} (2k)^n [2k\gamma - 2k + \gamma\lambda - \gamma][(1-\xi) a_{2k} + \xi b_{2k}]$$

$$= (1-\xi) \sum_{k=2}^{\infty} (2k)^n [2k\gamma - 2k + \gamma\lambda - \gamma] a_{2k}$$

$$+ \xi \sum_{k=2}^{\infty} (2k)^n [2k\gamma - 2k + \gamma\lambda - \gamma] b_{2k}$$

$$\le (1-\xi)\gamma(\alpha - \beta + \lambda - 1) + \xi\gamma(\alpha - \beta + \lambda - 1)$$

$$= \gamma(\alpha - \beta + \lambda - 1).$$

This implies that $h \in \varphi_n(\alpha,\beta,\gamma,\lambda)$.

Corollary 2.5: If $f_1(z)$, $f_2(z)$ are in $\varphi_n(\alpha,\beta,\gamma,\lambda)$ then the function defined by

$$g(z) = \frac{1}{2}[f_1(z) + f_2(z)]$$

is also in $\varphi_n(\alpha,\beta,\gamma,\lambda)$.

Theorem 2.6: Let $f_j(z) = z - \sum_{k=2}^{\infty} a_{2k,j} z^{2k}$ belongs to $\varphi_n(\alpha,\beta,\gamma,\lambda)$, for $j = 1, 2, ..., 2k$ and $0 < \lambda_j < 1$ such that $\sum_{j=1}^{2k} \lambda_j = 1$, then the function $F(z)$ defined by $F(z) = \sum_{j=1}^{2k} \lambda_j f_j(k)$ is also in $\varphi_n(\alpha,\beta,\gamma,\lambda)$.

Proof: For each $j \in \{1, 2, 3, ..., 2k\}$, we obtain

$$\sum_{k=2}^{\infty}(2k)^n[2k\gamma-2k+\gamma\lambda-\gamma]|a_{2k}|<\gamma(\alpha-\beta+\lambda-1)$$

$$F(z)=\sum_{j=1}^{2k}\lambda_j\left(z-\sum_{k=2}^{\infty}a_{2k,j}z^{2k}\right)$$

Since

$$=z-\sum_{k=2}^{\infty}\left(\sum_{j=1}^{2k}\lambda_j a_{2k,j}\right)z^{2k}$$

$$=\sum_{k=2}^{\infty}(2k)^n[2k\gamma-2k+\gamma\lambda-\gamma]\left[\sum_{j=1}^{2k}\lambda_j a_{2k,j}\right]$$

$$=\sum_{j=1}^{2k}\lambda_j\left[\sum_{k=2}^{\infty}(2k)^n[2k\gamma-2k+\gamma\lambda-\gamma]\right]$$

$$<\sum_{j=1}^{2k}\lambda_j\gamma(\alpha-\beta+\lambda-1)$$

$$<\gamma(\alpha-\beta+\lambda-1).$$

$$\therefore F(z)\in\varphi_n(\alpha,\beta,\gamma,\lambda).$$

Theorem 2.7: Let $f(z)\in\varphi_n(\alpha,\beta,\gamma,\lambda)$. Komato operator of f is well-defined by

$$k(z)=\int_0^1\frac{(c+1)^\phi}{\Gamma(\phi)}t^c\left(\log\frac{1}{t}\right)^{\phi-1}\frac{f(tz)}{t}dt,\quad c>-1,\phi\geq 0$$

then $k(z)\in\varphi_n(\alpha,\beta,\gamma,\lambda)$.

Proof: We have

$$\int_0^1 t^c\left(\log\frac{1}{t}\right)^{\phi-1}dt=\frac{\Gamma(\phi)}{(c+1)^\phi}$$

$$\int_0^1 t^{2k+c-1}\left(\log\frac{1}{t}\right)^{\phi-1}dt=\frac{\Gamma(\phi)}{(c+1)^\phi},\quad k=2,3,\ldots.$$

$$k(z)=\frac{(c+1)^\phi}{\Gamma(\phi)}\left[\int_0^1 t^c\left(\log\frac{1}{t}\right)^{\phi-1}zdt-\sum_{k=2}^{\infty}z^{2k}\int_0^1 a_{2k}t^{2k+c-1}\left(\log\frac{1}{t}\right)^{\phi-1}dt\right]$$

$$=z-\sum_{k=2}^{\infty}\left(\frac{c+1}{c+2k}\right)^\phi a_{2k}z^{2k}.$$

Since $f\in\varphi_n(\alpha,\beta,\gamma,\lambda)$ and since $\left(\frac{c+1}{c+2k}\right)^\phi<1$, we have

$$\sum_{k=2}^{\infty}(2k)^n[2k\gamma-2k+\gamma\lambda-\gamma]\left(\frac{c+1}{c+2k}\right)^\phi a_{2k}$$

$$<\gamma(\alpha-\beta+\lambda-1).$$

Theorem 2.8: Let $f\in\varphi_n(\alpha,\beta,\gamma,\lambda)$ then for every $0\leq\delta<1$ the function

$$\psi_\delta(z)=(1-\delta)f(z)+\delta\int_0^z\frac{f(t)}{t}dt.$$

Proof: We have

$$\psi_\delta(z) = z - \sum_{k=2}^{\infty}\left(1 + \frac{\delta}{2k} - \delta\right)a_{2k}z^{2k}.$$

Since, $\left(1 + \frac{\delta}{2k} - \delta\right) < 1, k \geq 2,$ so by Theorem 2.1

$$\sum_{k=2}^{\infty}\left(1 + \frac{\delta}{2k} - \delta\right)(2k)^n[2k\gamma - 2k + \gamma\lambda - \gamma]a_{2k}$$

$$< \sum_{k=2}^{\infty}(2k)^n[2k\gamma - 2k + \gamma\lambda - \gamma]a_{2k}$$

$$< \gamma(\alpha - \beta + \lambda - 1).$$

Therefore $\psi_\delta(z) \in \varphi_n(\alpha, \beta, \gamma, \lambda)$.

References

[1] Pall-Szabó, A.O.; Wanas, A.K. Coefficient estimates for some new classes of bi-Bazilevic functions of Ma-Minda type involving the Salagean integro-differential operator. Quaest. Math. 2021, 44, 495–502.
[2] Tang, H.; Khan, S.; Hussain, S.; Khan, N. Hankel and Toeplitz determinant for a subclass of multivalent q-starlike functions of order α. AIMS Math. 2021, 6, 5421–5439.
[3] Salagean G. S., Subclass of univalent functions, Lecture Notes inMathe., Springer Verlag, (1983), 362-372.
[4] Silverman H., Univalent functions with varying arguments, Houston Journal of Math., 7(2) (1981).
[5] Latha S. and Dileep L., A note on Salagean type functions, Global Journal of Mathematical Sciences : Theory and Practical, 2(1) (2010), 29-35.
[6] Eker S. S., Owa S., Certain classes of analyting functions involving Salagean operator, J. Inequal. Pure Appl. Math., 10(1) (2009), 1-22.
[7] Rahulkumar Katkade, Certain Subclasses of Analytic and Univalent Function by Ruscheweyh Derivative, International Journals of Pure and Engg. Maths (IJPEM) ISSN 0973-9424, Volume 1 No. I (Dec., 2013), pp.93-102.
[8] Rahulkumar Katkade, A Linear Operator of a New Class of Univalent Functions With Missing Second Coefficient, International Journals of Maths Sci. & Engg. Appl. (IJMSEA) ISSN 0973-9424, Vol. 8 No. II (March, 2014), pp. 257-269.
[9] Mostafa A. O., A study on starlike and convex properties for hypergeometric functions, JIPAM, 10 (Issue 3, Article 87) (2009), 8pp.