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# HERSCOVICI PEBBLING CONJECTURE FOR THE MIDDLE GRAPH OF FAN GRAPHS

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**Abstract.**
We prove that the Cartesian product of two connected graphs G and H satisfies the Herscovici's conjecture $f_{st}\ (G \times H) \ \leq \ f_s\ (G)\ f_t(H)$, for any positive integer s and t.

**Keywords:**
Herscovici's Conjecture, Lourdusamy conjecture, Graham conjecture, t-pebbling number, 2t-pebbling property. AMS Subject Classification: 05C76, 05C25, 05C38.

## Introduction

The concept of graph pebbling is used for the network related problems while transporting finite resources that are consumed in transit. It is an important problem for both theoretical and application perspectives. The concept of graph pebbling came to the literature form in order to solve the number theory problems. Lagarias and Sake first used to solve the number- theoretical conjecture of Erdos. Later Chung (1989) extended this concept and Hulbert (2004,2013) gave a detailed report on on the development of different areas in graph pebbling. The research on this area has been going on for the past 30 years. The aim of considering the connected and simple graph for our discussion is to consider all the possibility of reaching a product or the information to the destination by using minimum cost and time. A configuration of pebbles on a graph G is a function from V(G) to a set of non-negative integers. Throughout this paper, we consider a simple connected graph, G, having a configuration of pebbles. We define a pebbling move as: extracting a pair of pebbles from a vertex and inserting one of them on an adjacent vertex and eliminating the other pebble for the cost of the transportation. We also use the concept of the pebbling number, f(G), while proving the Herscovici conjecture. Lourdusamy et al. (2012,2022) defines the pebbling number of a vertex v in a graph G. It is the smallest number f(G, v) that allows us to shift a pebble to v by doing the series of pebbling moves, regardless of where these pebbles are located on G's vertices. Lourdusamy et al(2022) defines the pebbling number of G. It is denoted as f(G) and it is the largest of the pebbling numbers of the vertices of G.
Instead of moving one pebble to a target if we desire to move t pebbles, we get the concept of t-pebbling number.
Ye et al.(2014) defines the t-pebbling number $f_t$(G) of G. It is the maximum of the t pebbling numbers of the vertices of G. Let p and q denote the number of pebbles present on the vertices of G and number of occupied vertices of G respectively. The graph G satisfies the 2t pebbling

property if we could transfer 2t pebbles to the destination for p ≥ $2f_t$(G)- q + 1. Considering the pebbling number of a graph and 2t-pebbling property we could prove the Herscovici conjecture for all the possible combination of pebble distribution on Cartesian product of two connected graphs. This determines the total cost of transportation to reach the destination if the infrastructure or the road map is of the Cartesian product of two connected graphs. The aim of this paper is to deal with Herscovici pebbling conjecture for the middle graph of fan graphs.

## 2 Preliminaries

The following theorems and definitions are used to prove the theorems in section 3.
**Definition 1.** Let G be a graph. We construct a new graph from G whose vertex set is the set of vertices in G and new vertices which are inserted into every edge of G and whose edge set is the set of all edges in G plus edges obtained by joining those pairs of new vertices which lie on adjacent edges of G. The graph constructed is middle graph of G and denoted by M(G).
**Definition 2.** Consider the path $P_{n-1}$: $x_1$,…, $x_{n-1}$. Then add an extra vertex $x_0$ to the path $P_{n-1}$ and connect it to all of the path's vertices. The graph obtained by this process is called fan graph and it is denoted by $F_n$.

**Theorem 1.** (Lourdusamy et al (2022)). For a path $P_n$, $f_t$ ($P_n$) =$t2^{n-1}$.
**Definition 3.** (Herscovici, D. S., and Higgins, A. W. (1998)).

Let V (G× H) = V (G) ×V (H). The edge set of G×H is defines as: E(G×H)={$\big((x,y),(x',y')\big): x = x'\ and\ (y,y') \in E(H) or\ (x,x') \in E(G)\ and\ y = y'$}. The graph G× H is called the Cartesian product of G and H. We denote $p_i$ and $q_i$ for the number of pebbles and the number of occupied vertices on $x_i$ × H respectively.

**Notation 1.** Let p(x) stands for the total number of pebbles on the vertex x. The notation $A\xrightarrow{t}B$ refers to moving t pebbles from the vertices of set A to that of the set B using pebbling moves.

The aim of this paper is to verify the Herscovici conjecture for the Cartesian product of middle graph of fan graphs. Herscovici conjecture evolved from Graham Conjecture and Lourdusamy Conjecture. The first conjecture is f (G ×H) ≤ f (G) ×f (H). The second conjecture is $f_t$ (G× H) ≤ f (G) $f_t$ (H). From these two conjectures we get Herscovici conjecture: $f_{st}\ (G \times H) \leq f_s\ (G)\ f_t(H)$.

**Theorem 2. (**Ye et al. (2014)). For n ≥4, f(M($F_n$)) = 3n-1.
**Corollary 1.** For n≥ 4; $f_t$(M ($F_n$) )= (8t + 3n- 9):
**Proof.** Follows by induction on t and by Theorem 2.3.

**Theorem 3. (**Ye et al. (2014)). M($F_n$) satisfies the 2- pebbling property.
**Corollary 2.** M($F_n$) satisfies the 2t- pebbling property.
**Proof.** Follows by induction on t and by Theorem 2.3 and Theroem 2.4.

## 3. Herscovici Pebbling Conjecture on M($F_n$) ×M($F_m$)

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**Theorem 4.** If H satisfies the 2t-pebbling property, then for n, m ≥ 3; $f_{st}$ (M($F_n$) ×M($F_m$)) ≤ $f_s$ (M($F_n$)) $f_t$ (M($F_m$) )= (8s + 3n-9)(8t + 3n- 9).

**Proof**. Let G = M($F_n$) and H = M($F_m$). Let V (G) = $\{x_0, x_i, x_{j(j+1)}, x_{0i} : 1\leq i \leq n-1, 1 \leq j \leq n-2\}$ and V (H) =$\{y_0, y_i, y_{j(j+1)}, y_{0i} : 1\leq i \leq n-1, 1 \leq j \leq n-2\}$. First let us try to shift t pebbles to the desired vertex by induction on t using the Lourdusamy pebbling conjecture and then shift (s-1)t pebbles to the desired vertex using induction on s. Let D be a distribution of (8s+3n-9) $f_t$(M ($F_m$) ) pebbles on the vertices of M($F_n$) ×M($F_m$).

**Case 1**: Let $(x_0, y)$ be the target.
Let y be the vertices from H. If $p_0 \geq f_t$ (H), then we could put t pebbles on $(x_0, y)$. So let $p_0 < f_t$ (G). If there is k ∈ {1,2, ….. n-1} such that $\frac{p_{0k}+ q_{0k}}{2} > f_t$ (H), then we can transfer 2t pebbles to $(x_{0k}, y)$ and t pebbles to $(x_0, y)$. If there is i ∈{1,2, ….. n-1} such that $\frac{p_i+ q_i}{2} > 2f_t$ (H), then we can transfer 4t pebbles to $(x_i, y)$. Further it is easy to see that $(x_i, y) \xrightarrow{2t} (x_{0k}, y \xrightarrow{t} (x_0, y)$. Thus, we can transfer t pebbles to $(x_0, y)$. If there exists j ∈{1,2, ….. n-2} with $\frac{p_{i(j+1)}+ q_{i(j+1)}}{2}$ >$2f_t$ (H), then we transfer 4t pebbles to ( $x_{i(j+1),}y$). Further it follows that $(x_{i(j+1)}, y) \xrightarrow{2t} (x_{0k}, y) \xrightarrow{t} (x_0, y)$. Suppose $p_0 < f_t$ (G), $\frac{p_{0k}+ q_{0k}}{2} \leq f_t$ (H) and $\frac{p_i+ q_i}{2} \leq 2\, f_t$ (H). Then we have one of the following equations:

$$p_0 + \sum_{k=0}^{n-1} \frac{p_{0k}- q_{0k}}{2} \geq f_t(\text{H}) \quad (1)$$

$$p_0 + \sum_{i=0}^{n-1} \frac{p_i- q_i}{4} \geq f_t(\text{H}) \quad (2)$$

$$p_0 + \sum_{j=0}^{n-1} \frac{p_{i(j+1)}- q_{i(j+1)}}{4} \geq f_t(\text{H}) \quad (3)$$

$$p_0 + \sum_{k=0}^{n-1} \frac{p_{0k}- q_{0k}}{2} + \sum_{i=0}^{n-1} \frac{p_i- q_i}{4} + \sum_{j=0}^{n-1} \frac{p_{i(j+1)}- q_{i(j+1)}}{4} \geq f_t(\text{H}) \quad (4)$$

Thus, we could shift t pebbles to $(x_0, y)$. The number of pebbles remainning on the graph is (8s+3n-9) $f_t$ (H)- $f_t$ (H) ≥ (8s+3n-10) $f_t$ (H) = $f_{s-1}$ (G) $f_t$ (H). Thus we could put (s-1)t more pebbles on the destination. Thus the destination gets st pebbles and we are done.

**Case 2:** Let $(x_i, y)$ or $(x_{i(j+1)}, y)$ be the target vertex, where $1\leq i \leq n-1, 1 \leq j \leq n-2$.
Without loss of generality, we assume $(x_1, y)$ to be the target. We consider 3n-3 copies of M($F_m$). If $p_1 \geq f_t$ (H), then we can put t pebbles to $(x_1, y)$. We assume $p_1 < f_t$ (G). If there is $\frac{p_{01}+ q_{01}}{2} > f_t$ (H), then we can put 2t pebbles to $(x_{01}, y)$ and then put t pebble on $(x_1, y)$. Similarly, we can transfer from $p_{12}$. If there exists some k ∈ {2, ….. n-1} with $\frac{p_{0k}+ q_{0k}}{2} > 2f_t$ (H), then we can transfer 4t pebbles to $(x_{0k}, y)$. Further we see that $(x_{0k}, y) \xrightarrow{2t} (x_{01}, y) \xrightarrow{t} (x_0, y)$. Similarly, we can transfer from $p_2, p_{23}$ and $p_0$. For some $i$ ∈ {4,….. n-1} with $\frac{p_{0k}+ q_{0k}}{2} > 4f_t$ (H), we could transfer 8t pebbles to $(x_i, y)$. Further it follows that $(x_i, y) \xrightarrow{4t} (x_{0k}, y) \xrightarrow{2t} (x_{01}, y) \xrightarrow{t} (x_1, y)$. Thus, we can transfer t

pebbles to $(x_1, y)$. If there exists some $j \in \{3,4,\ldots\ n\text{-}2\}$ with $\frac{p_{i(j+1)}+ q_{i(j+1)}}{2} > 4f_t$ (H), then we can transfer 8t pebbles to $(x_{i(j+1)}, y)$. Clearly $(x_{i(j+1)}, y) \xrightarrow{4t} (x_{0k}, y) \xrightarrow{2t} (x_{01}, y) \xrightarrow{t} (x_1, y)$. If there exists some $\frac{p_0+ q_0}{2} > 2f_t$ (H), then we can transfer 4t pebbles to $(x_0, y)$. Clearly $(x_0, y) \xrightarrow{2t} (x_{01}, y) \xrightarrow{t} (x_1, y)$. Thus, we can transfer t pebbles to $(x_1, y)$. Similarly, we can transfer t pebbles by considering $p_2$. Suppose $p_1 < f_t$ (G); $\frac{p_{0k}+ q_{0k}}{2} \le 2f_t$ (H), $\frac{p_i+ q_i}{2} \le 4f_t$ (H) and $\frac{p_0+ q_0}{2} \le 2f_t$ (H) then we have one of the following equations:

$$p_1 + \sum_{k=0}^{n-1} \frac{p_{0k}- q_{0k}}{4} + \frac{p_{01}- q_{01}}{2} \ \ge\ f_t \text{ (H)} \quad (6)$$

$$p_1 + \sum_{i=3}^{n-1} \frac{p_i- q_i}{8} + \sum_{k=2}^{n-1} \frac{p_{0k}- q_{0k}}{4} + \frac{p_{01}- q_{01}}{2} \ \ge\ f_t \text{ (H)} \quad (7)$$

$$p_1 + \sum_{j=3}^{n-2} \frac{p_{i(j+1)}- q_{i(j+1)}}{8} + \sum_{k=2}^{n-1} \frac{p_{0k}- q_{0k}}{4} + \frac{p_{01}- q_{01}}{2} \ \ge\ f_t \text{ (H)} \quad (8)$$

$$p_1 + \frac{p_{12}- q_{12}}{2} \ or\ p_1 + \frac{p_{01}- q_{01}}{2} \ge\ f_t \text{ (H)} \quad (9)$$

$$p_1 + \sum_{k=0}^{n-1} \frac{p_{0k}- q_{0k}}{4} + \sum_{i=3}^{n-1} \frac{p_i- q_i}{8} + \sum_{j=3}^{n-2} \frac{p_{i(j+1)}- q_{i(j+1)}}{8} + \frac{p_{01}- q_{01}}{2} + \frac{p_{12}- q_{12}}{2} + \frac{p_{23}- q_{23}}{2} \ge\ f_t \text{ (H)} \quad (10)$$

We can put t pebbles to $(x_1, y)$. Thus, the remaining pebbles on the graph is (8s+3n-9) $f_t$ (H)- $f_t$ (H) ≥ (8s+3n-10) $f_t$ (H) = $f_{s-1}$ (G) $f_t$ (H). So we can shift (s-1)t more pebbles to the destination and we are done.

**Case 3:** Let the destination vertex be $(x_{0k}, y)$ where $j \in \{1,2,\ldots\ n\text{-}1\}$.
Without loss of generality, we assume that $(x_{01}, y)$ be the destination vertex. If $p_{01} \ge\ f_t$ (H), then t pebbles can be shifted to $(x_{01}, y)$. We assume $p_{01} < f_t$ (G). If there exists some k ∈ {2, ….. n-1} with $\frac{p_{0k}+ q_{0k}}{2} > f_t$ (H), then we can transfer 2t pebbles to $(x_{0k}, y)$ and t pebbles to $(x_{01}, y)$. Similarly, we can transfer from $p_1, p_{12}$ and $p_0$. If there exists some i ∈ {2, ….. n-1} with $\frac{p_i+ q_i}{2} > 2f_t$ (H), then 4t pebbles are transferred to $(x_i, y)$. Similarly, we can transfer from $p_1, p_{12}$ and $p_0$. $(x_i, y) \xrightarrow{2t} (x_{0k}, y)(k\ 1) \xrightarrow{t} (x_{01}, y)$. Thus, t pebbles can be shifted to $(x_{01}, y)$. If there exists some $j \in \{2,4,\ldots$ n-2} with $\frac{p_{i(j+1)}+ q_{i(j+1)}}{2} > 2f_t$ (H), then we can transfer 4t pebbles to $(x_{i(j+1)}, y)$. It is clear that $(x_{i(j+1)}, y) \xrightarrow{2t} (x_{0k}, y)(k \ne 1) \xrightarrow{t} (x_{01}, y)$. Suppose $p_{01} < f_t(G)$, $\frac{p_{0k}+q_{0k}}{2} \le f_t(H)$, $\frac{p_i+q_i}{2} \le 2f_t(H) \frac{p_{j(j+1)}+q_{j(j+1)}}{2} < 2f_t(H), \frac{p_1+q_1}{2} < f_t(H), \frac{p_0+q_0}{2} < f_t(H),\ nd\ \frac{p_{12}+q_{12}}{2} < f_t H$. Then we have one of the following equation:

$$p_{01} + \sum_{k=2}^{n-1} \frac{p_{0k}-q_{0k}}{2} \ge f_t(H) \quad (11)$$

$$p_{01} + \sum_{i=2}^{n-1} \frac{p_i-q_i}{4} + \sum_{k=2}^{n-1} \frac{p_{0k}-q_{0k}}{2} \ge f_t(H) \quad (12)$$

$$p_{01} + \sum_{j=2}^{n-1} \frac{p_{j(j+1)}-q_{j(J+1)}}{4} + \sum_{k=2}^{n-1} \frac{p_{0k}-q_{0k}}{2} \ge f_t(H) \quad (13)$$

$$p_{01} + \sum_{k=2}^{n-1} \frac{p_{0k}-q_{0k}}{2} + \sum_{i=1}^{n-1} \frac{p_i - q_i}{4} + \sum_{j=1}^{n-1} \frac{p_{j(j+1)} - q_{j(J+1)}}{4} + \frac{p_0 - q_0}{2} \geq f_t(H) \quad (14)$$

We could transfer \$t\$ pebbles to $(x_{01}, y)$. Then the total number of pebbles remaining on the graph is (8s+3n-10) $f_t$ (H)- $f_t$ (H) ≥ (8s+3n-10) $f_t$ (H) = $f_{s-1}$ (G) $f_t$ (H). Thus (s-1)t more pebbles to the destination can be moved and so we are done.

4 **Conclusion**

This article proved the Herscovisi Conjecture for two middle graphs of fan graphs. It is a kind of a network graph where we need to transfer the information or transport an item to the target. we took the graph theory approach to find the minimum cost require to do so. For all these operations we used the shortest path possible and for all combination of pebble distributions. This helps us for memory allocation in the computer, based on the available information. We have a lot scope to explore all the conjectures for the network related graphs.

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