Industrial Engineering Journal
ISSN: 0970-2555
Volume : 52, Issue 7, July : 2023

ROLE OF COMMUTATIVE RINGS IN MULTIPLICATION OPERATION

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Abstract
Commutative rings is a branch of abstract Algebra that deals with the multiplication oper- ation. It is the system that admits well-behaved operations of addition and multiplication. In the last two decades, commutative algebraist has shown very much interest in the study of commutative ring extensions with the same nonzero identity. The aim is to study the ring theoretic properties of intermediate rings of a ring extension. Many well known ring theoretic properties have been studied on intermediate rings like valuation, Proofer, etc. Many topological properties were also studied on ring extensions.

Historical Background and Basic Results

If $R \subset T$ is a ring extension of commutative rings, then it is assumed that R and T have same nonzero unity. By an intermediate ring (resp., proper intermediate ring), we mean a subring (resp., proper subring) of T containing R (resp., properly containing$R$). The study of intermediate rings of a ring extension was first started by Ferrandand Olivier in 1970. They considered all those rings extensions where there is no proper intermediate ring and named them minimal ring extensions. Thus, a ring extension $R \subset T$ is a minimal ring extension if there is no proper intermediate ring between R *and* T. In 1972 and 1978, Modica and Dechéne in recalled these ring extensions in their unpublished thesis and used the terms maximal subring and adjacent extension, respectively. Since then, minimal ring extension became the famous topic of commutative ring theory in 20th and 21st century and algebraist started working on this. In the starting of 21st century, the main question for the algebraist was the existence of a minimal ring extension for arbitrary commutative ring which was solvedby Dobbs in 2006. Ferrand and Olivier classified the minimal ring extensionsof a field. Following this, Dobbs and many algebraist worked on classifying the minimal

ring extensions of an arbitrary domain and certain commutative rings. If $R \subset T$ is a minimal ring extension, then R is called a maximal subring of T. Note that not every ring has a maximal subring, for example, take Z. Azarang et al.
studied the rings for which maximal subrings exist. In 1974, Gilmer and Huckaba generalized the concept of minimal ring extensions by introducing the concept of ∆-extension of rings. A ∆-extension of rings is a ring extension where the sum of any two intermediate rings is again an intermediate ring. The studyof intermediate rings of a ring extension became more famous when Gilbert in 1994 introduced a generalization of minimal ring extensions. Motivated by the concept of ∆- extension of rings, Gilbert in his unpublished thesis introduced λ-extension of rings, a ring extension where the set of intermediate rings is linearly ordered by inclusion. Thus, we have the following implications:

Minimal ring extension ⇒ λ-extension of rings ⇒ ∆-extension of rings

However, the reverse implications are not true. For example, $Q \subset Q(2^{1/4})$ is a λ- extension of rings but not minimal, and $Z \subset Z[1/2, 1/3]$ is a ∆-extension of rings but not a λ-extension of rings.
The study of intermediate rings found more interest when algebraist started workingon certain

Industrial Engineering Journal
ISSN: 0970-2555
Volume : 52, Issue 7, July : 2023

ring theoretic properties which are not satisfied by a proper subring R of a domain T but every proper intermediate domain including T satisfies these properties. The study of these properties and extension of integral domains started by Visweswaran in 1990, see [96]. He found a non Noetherian subring of a Noetherain domain such that each intermediate ring is Noetherian and these subrings are called maximal non Noetherian subrings. Then algebraist started working on maximal non P-subrings ofa domain for well known ring theoretic properties P in the literature, namely P=: PID,

Prüfer, valuation, pseudo-valuation, Jaffard, ACCP etc., One more concept of intermediate rings found attention when Badawi in 1999 extended the concept of integral domains to rings in which the set of all nilpotent elements is a prime ideal and comparable to every ideal of that ring. He named them φ-rings. Since then, φ-rings become the centre of attraction for many commutative algebraist. Badawi and his co-authors extended many well known concepts in the literature. They defined φ-chained rings, φ-pseudo-valuation rings, φ-Prüfer rings, φ-Dedekind rings, etc. which are generalizations of well known integral domains like valuation domains, pseudo-valuation domains, Prüfer domains, Dedekind domains, etc.

In this chapter, we discuss the existing literature on commutative ring theory and ring extensions of commutative rings which are used throughout the thesis. We recall the existing results and standard terminology from commutative ring extensions. We also give a summary of the thesis.

Throughout the thesis, it is assumed that all rings are commutative with nonzero identity; all ring extensions, ring homomorphisms, and algebra homomorphisms are unital. The symbol $\subseteq$ is used for inclusion and $\subset$ used for proper inclusion. Let R and T be rings. Then by $R \subseteq T$ and $R \subset T$, we mean that R is a subring of T and R is a proper subring of T, respectively. For any ring (resp., domain) R, let tq(R) (resp., qf(R)) denotes the total quotient ring (resp., the quotient field) of R, dim(R) denotes the Krull dimension of R, R' denotes the integral closure of R in tq(R). The spectrum (resp., maximal spectrum) of a ring R, denoted by Spec(R) (resp., Max(R)), is the set of all prime (resp., maximal) ideals of R. We use Nil(R) to denotes the set of all nilpotent elements and Z(R) to denotes the set of all zero-divisors of R. For an ideal a, Rad(a) denotes the radical of a. All the elements of $R \setminus$ Z(R) are said to be regular elements of R and an ideal is said to be regular if it contains a regular element. By an overring (resp., proper overring) of R, we mean any subring of tq(R) which contains R (resp., properly). For any ring extension $R \subset T$, the conductor $(R : T)$ is the set $\{t \in T \mid tT \subseteq R\}$, $\mathrm{Rad}_R(R : T)$ denotes the radical of $(R : T)$ in R, and $[R, T]$ denotes the set of all subrings of T which contains

R. By a local ring (resp., semi-local ring), we mean a ring with a unique maximal ideal

(resp., finitely many maximal ideals). As usual, if E is an R-module, then E_p denotes the localization of E on a prime ideal p of R and Supp(E) denotes the set of prime ideals p of R such that $E_p \neq 0$.

For any ideals a, b of R, $(a : b) = \{x \in \mathrm{tq}(R) \mid xb \subseteq a\}$. For

further study.

We start with the concept of a minimal ring extension which was initiated by Ferrand and Olivier.

DEFINITION 1.1.1. Let $f : R \hookrightarrow T$ be an injective ring homomorphism that is not an isomorphism. Then f is called a minimal ring homomorphism if any factorization $f = g \circ h$ of f produce that one of the ring homomorphisms g, h is an isomorphism. Let R be any proper subring of a ring T. Then T is called a minimal ring extension of R or equivalently, R is a maximal subring of T if the inclusion map $R \hookrightarrow T$ is a minimal ring homomorphism, that is, if $|[R, T]| = 2$ (as usual, $||$ is used for cardinality).

By a minimal overring of R, we mean any overring of R which is a minimal ring extension of R. It is easy to see that if $R \subset T$ is a minimal ring extension, then either it is an integral extension or R is integrally closed in T. If the latter holds, then $f : R \hookrightarrow T$ is a flat epimorphism, by [Théorème 2.2]. Note that by [Théorème 2.2(i)] and [Lemme 1.3], a ring extension $R \subset T$ is a minimal ring extension if and only if there exists a unique maximal ideal q of R such that $R_q \hookrightarrow T_q := T_{R \setminus q}$ is not an isomorphism; moreover, $R_q \hookrightarrow T_q$ is then a minimal ring extension, and $R_p \hookrightarrow T_p$ is an isomorphism for all $p \in \text{Spec}(R) \setminus \{q\}$. The maximal ideal q appearing in the above statement is called the crucial maximal ideal, see [Definition 2.9].

In Dobbs proved that every ring R has a ring extension T such that $R \subset T$ is a minimal ring extension. Azarang called this ring R a maximal subring of T. Note that not every ring has a maximal subring, for example, take Z. Azarang et al. studied fields, domains and rings for which maximal subrings exist.

The concept of minimal ring extensions generalized by Gilmer and Huckaba. They introduced the concept of Δ-extension of rings as follows:

DEFINITION 1.1.2. A ring extension $R \subseteq T$ is said to be a Δ-extension of rings or R is called a Δ-subring of T if $[R, T]$ is closed under addition. Moreover, if $T = \text{tq}(R)$, then R is called a Δ-ring. Throughout the thesis, by Δ-extension we mean Δ-extension of rings.

Gilbert introduced the concept of λ-extension of rings which was motivated by the concept of minimal ring extensions and Δ-extensions. The definition of λ-extension of rings is as follows:

DEFINITION 1.1.3. A ring extension $R \subseteq T$ is said to be a λ-extension of rings or R is called a λ-subring of T if $[R, T]$ is linearly ordered under inclusion. Moreover, if R is a domain and $T = \text{qf}(R)$, then R is called a λ-domain. Throughout the thesis, by λ-extension we mean λ-extension of rings.

In [cf. Nagata, 1962, p.2], Nagata introduced the concept of idealization of a module. For a ring R and an R-module E, the idealization $R(+)E$ is the ring defined as follows: its additive structure is that of the abelian group $R \oplus E$, and its multiplication is defined by

$(r_1, e_1)(r_2, e_2) := (r_1r_2, r_1e_2 + r_2e_1)$ for all $r_1, r_2 \in R$ and $e_1, e_2 \in E$. It will be convenient to view R as a subring of $R(+)E$ via the canonical injective ring homomorphism that sends r to $(r, 0)$. Now, we recall some basic results on the idealization of a module.

RESULT 1.1.4. [2, Theorem 3.1] Let R be a ring, a be an ideal of R, E be an R-module and F be a submodule of E. Then $a(+)F$ is an ideal of $R(+)E$ if and only if $aE \subseteq F$.

Moreover, if $a(+)F$ is an ideal of $R(+)E$, then $(R(+)E)/(a(+)F) \cong (R/a)(+)(E/F)$.

RESULT 1.1.5. Let R be a ring and E be an R-module. Then the following hold:

(i) $\text{Max}(R(+)E) = \{m(+)E : m \in \text{Max}(R)\}$, see [2, Theorem 3.2(1)].

(ii) $\text{Spec}(R(+)E) = \{m(+)E : m \in \text{Spec}(R)\}$ and so $\dim(R(+)E) = \dim(R)$, see [66, Theorem 25.1(3)].

(iii) $\text{Nil}(R(+)E) = \text{Nil}(R)(+)E$, see [2, Theorem 3.2(3)].

RESULT 1.1.6. [66, Theorem 25.3] If R is a ring and E is an R-module, then $Z(R(+)E) = \{(r, e) : r \in Z(R) \cup Z(E), e \in E\}$ where $Z(E) = \{r \in R$: there exists nonzero $e \in E$ such that $re = 0\}$.

RESULT 1.1.7. [Corollary 25.5] Let R be a ring and E be an R-module. Then the following hold:

(i) $\text{tq}(R(+)E) \cong R_S(+)E_S$, where $S = R \setminus (Z(R) \cup Z(E))$.

(ii) $(R(+)E)_{p(+)E} \cong R_p(+)E_p$ for prime ideal p of R.

(iii) If $Z(E) \subseteq Z(R)$, then $\text{tq}(R(+)E) \cong \text{tq}(R)(+)E_S$, where $S = R \setminus (Z(R) \cup Z(E))$.

Industrial Engineering Journal
ISSN: 0970-2555
Volume : 52, Issue 7, July : 2023

RESULT 1.1.8. [2, Theorem 3.3(4)] Let R be a domain and E be a divisible R-module.Then every nonzero ideal of $R(+)E$ is of the form $a(+)E$, where a is an ideal of R.

RESULT 1.1.9. [39, Lemma 2.3] Let R be a ring and E be an R-module. Then $R[(r, m)] = R(+)Rm$ for all $r \in R$ and for all $m \in E$.

RESULT 1.1.10. [39, Remark 2.9] Let R be a ring and E be an R-module. Then any R-subalgebra of $R(+)E$ is of the form $R(+)F$ for some R-submodule F of E.

For a ring R, Aut(R) denotes the set of all automorphisms of R. Let $R \subset T$ be a ring extension. Throughout the thesis, we assume that G is a subgroup of Aut(T) such that $\sigma(R) \subseteq R$ for all $\sigma \in G$, unless otherwise stated. We denote the orbit of $t \in T$ under $G$by $\theta_t = \{\sigma(t) : \sigma \in G\}$. We call that G is locally finite if θ_t is finite, for all $t \in T$, and if $G$is locally finite, then we define $\tilde{t} := \prod_{t_i \in \theta_t} t_i$. By T^G, we mean the set of all elements of$T$ left fixed by every element of G. Obviously, $\tilde{t} \in T^G$, for all $t \in T$. Let R (resp, $R \subseteq T$) satisfies a ring theoretic property P. We say that P is a G-invariant property if R^G (resp,
$R^G \subseteq T^G$) also satisfies P. Now, we recall some results on group invariant which are used frequently in this thesis.

RESULT 1.1.11. [49, Lemma 2.1(b)] Let (R, m) be a local ring and G be a subgroup ofAut(R). Then (R^G, m^G) is a local ring.

RESULT 1.1.12. [49, Lemma 2.2] Let R be a ring and G be a subgroup of Aut(R). If G is locally finite, then $R^G \subseteq R$ is an integral extension.

Let R be a domain and G be a subgroup of Aut(R). Then G can be extended to qf(R) via $\sigma(r/s) = \sigma(r)/\sigma(s)$ for all $\sigma \in G$, $r \in R$ and nonzero $s \in R$.

RESULT 1.1.13. [49, Lemma 2.3] Let R be a domain and G be a subgroup of Aut(R). If G is locally finite, then qf$(R^G) = (\text{qf}(R))^G$.

DEFINITION 1.1.14. Let R be an integral domain. Then R is said to be a valuation domainif for each nonzero $x \in$ qf(R), we have $x \in R$ or $x^{-1} \in R$.

RESULT 1.1.15. [49, Proposition 2.7] Let R be a valuation domain and G be a subgroupof Aut(R). Then R^G is a valuation domain.

RESULT 1.1.16. [95, Lemma 2.4] Let $R \subset T$ be a ring extension, $|G|$ be finite and a unit of T, and let $t \in T^G$ be nonzero. If $t = \sum_{i=0}^{n} r_i x_i$ for some $r_i \in R$ and $x_i \in T^G$, then $t = |G|^{-1}(\sum_{i=0}^{n} t_i x_i)$ for some $t_i \in R^G$.

Let P be a ring theoretic property and $R \subset T$ be a ring extension. Then (R, T) is said to be a P-pair if each ring in $[R, T]$ satisfies the property P. Also, R is a maximal non P-subring of T if R does not satisfy the property P but each ring in $[R, T] \setminus \{R\}$ satisfies the property P.

DEFINITION 1.1.17. [74] A domain R is said to be a Prüfer domain if each finitely gener- ated ideal of R is invertible. A well known characterization of Prüfer domain is as follows: A domain R is said to be a Prüfer domain if and only if R_p is a valuation domain for each $p \in$ Spec(R) if and only if R_m is a valuation domain for each $m \in$ Max(R), see [74, Theorem 64]. For example, Z is a Prüfer domain. Note that every overring of a Prüfer domain R is an intersection of localizations of R on prime ideals of R.

DEFINITION 1.1.18. [74] A domain R is said to be a Bézout domain if each finitely gener- ated ideal of R is principal. It follows that a semi-local Prüfer domain is a Bézout domain, see [74, Theorem 60].

For a ring extension $R \subset T$ of domains, R is said to be a maximal non valuation subring of T if R is not a valuation domain but each ring in $[R, T] \setminus \{R\}$ is a valuation domain, see [91]. Similarly, R is

said to be a maximal non Prüfer subring (resp., maximal non integrally closed subring) of T if R is not a Prüfer domain (resp., not integrally closed) but each ring in $[R, T] \setminus \{R\}$ is a Prüfer domain (resp., integrally closed domain).
The next two definitions generalizes the definition of valuation domain.

(1) Let $R \subset T$ be a ring extension of integral domains. Then R is said to be a valuation subring of T (R is a VD in T, for short), if whenever nonzero $x \in T$, we have $x \in R$ or $x^{-1} \in R$, see [10].

(2) An integral domain R is said to be a valuative domain, if for each nonzero $x \in \mathrm{qf}(R)$, $R \subseteq R[x]$ or $R \subseteq R[x^{-1}]$ has no proper intermediate ring, see [33].

A divided prime ideal is a prime ideal q of a ring R such that $qR_q = q$, see [37]. Badawi characterized the divided prime ideal of R. *a*s a prime ideal which is comparable to every ideal of R. In [21], Badawi introduced the class

H = $\{R \mid R$ is a commutative ring and $\mathrm{Nil}(R)$ is a divided prime ideal of $R\}$

In [3], Anderson and Badawi used the notation H_0 to denote the subset of H such that $\mathrm{Nil}(R) = \mathrm{Z}(R)$. For $R \in \mathrm{H}$, Badawi in [21] considered a ring homomorphism φ from $\mathrm{tq}(R)$ to $R_{\mathrm{Nil}(R)}$ given by $\varphi(r/s) = r/s$ for $r \in R$ and $s \in R \setminus \mathrm{Z}(R)$. Note that the restriction of φ to R is also a ring homomorphism given by $\varphi(r) = r/1$ for $r \in R$. Thus, we also conclude that $\varphi(R) = R$ for $R \in \mathrm{H}_0$. We now list some results on φ-rings which are already in literature and are frequently used in this thesis. Note that the first five results are proved in [21] whereas the last one is proved in [4]. Let $R \in \mathrm{H}$. Then

(a) $\varphi(R) \in \mathrm{H}_0$.
(b) $\mathrm{Ker}(\varphi) \subseteq \mathrm{Nil}(R)$.
(c) $\mathrm{Nil}(\mathrm{tq}(R)) = \mathrm{Nil}(R)$.
(d) $\mathrm{Nil}(R_{\mathrm{Nil}(R)}) = \varphi(\mathrm{Nil}(R)) = \mathrm{Nil}(\varphi(R)) = \mathrm{Z}(\varphi(R))$.
(e) $\mathrm{tq}(\varphi(R)) = R_{\mathrm{Nil}(R)}$ is a local ring with maximal ideal $\mathrm{Nil}(\varphi(R))$, and $R_{\mathrm{Nil}(R)}/\mathrm{Nil}(\varphi(R)) = \mathrm{tq}(\varphi(R))/\mathrm{Nil}(\varphi(R)) = \mathrm{qf}(\varphi(R)/\mathrm{Nil}(\varphi(R)))$.
(f) $(R/\mathrm{Nil}(R))' = R'/\mathrm{Nil}(R)$ provided $R \in \mathrm{H}_0$.

We have the following observations on φ-rings:

RESULT 1.1.19. [3, Lemma 2.5] Let $R \in \mathrm{H}$ and p be a prime ideal of R. Then R/p is ring isomorphic to $\varphi(R)/\varphi(p)$.

RESULT 1.1.20. Let $R \in \mathrm{H}$. Then R is local if and only if $\varphi(R)$ is local.

PROOF. Let R be local with a unique maximal ideal m. Then $\varphi(m)$ is a maximal ideal of $\varphi(R)$, by Result 1.1.19. Now, if n is any maximal ideal of $\varphi(R)$, then $n = \varphi(p)$ for some prime ideal p of R. Then p is a maximal ideal of R by Result 1.1.19. It follows that$p = m$. Consequently, $n = \varphi(m)$. Thus, $\varphi(m)$ is a unique maximal ideal of $\varphi(R)$.

Conversely, assume that $\varphi(R)$ is local with a unique maximal ideal n. Then $n = \varphi(p)$ for some maximal ideal p of R. Now, if m is any maximal ideal of R, then $\varphi(m)$ is a maximal ideal of $\varphi(R)$, by Result 1.1.19. It follows that $\varphi(m) = \varphi(p)$. Since $\mathrm{Ker}(\varphi) \subseteq \mathrm{Nil}(R)$ by (b), $\mathrm{Ker}(\varphi)$ contained in m and p. Thus, $m = p$ and hence p is a unique maximal ideal of R.

RESULT 1.1.21. Let $R \in \mathrm{H}$. Then every overring of R is in H.

PROOF. Let T be an overring of R. Then $\mathrm{Nil}(T) = \mathrm{Nil}(R)$ by (c). We assert that $\mathrm{Nil}(R)$ is a prime ideal of T. Let $a/b, c/d \in T$ for $a, c \in R$ and $b, d \in R \setminus \mathrm{Z}(R)$. If $(a/b)(c/d) \in \mathrm{Nil}(T) = \mathrm{Nil}(R)$, then $ac/bd = r/1$ for some $r \in R$ such that $r^n = 0$ for some $n \in \mathrm{N}$. It follows that $(ac)^n = 0$. Since $R \in \mathrm{H}$, $\mathrm{Nil}(R)$ is a prime ideal of R. Consequently, $a \in \mathrm{Nil}(R)$ or $c \in \mathrm{Nil}(R)$. Therefore, $a/b \in \mathrm{Nil}(T) = \mathrm{Nil}(R)$ or$c/d \in \mathrm{Nil}(T) = \mathrm{Nil}(R)$. Thus, $\mathrm{Nil}(R)$ is a prime ideal of T. It remains to show that$\mathrm{Nil}(R)$ is a divided ideal of T. Let a be a proper ideal of T and let $b = a \cap R$. Since

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ISSN: 0970-2555
Volume : 52, Issue 7, July : 2023
Nil(R) is a divided ideal of R, Nil(R) ⊆ b or b ⊆ Nil(R). If Nil(R) ⊆ b, then Nil(R) ⊆ $a$as b ⊆ a. Let b ⊆ Nil(R). We claim that a ⊆ Nil(R). Let $a/b \in a$ for some $a \in R$ and $b \in R \setminus Z(R)$. Then $a \in b$. It follows that $a \in$ Nil(R) and hence $a/b \in$ Nil(T) = Nil(R). Thus, a ⊆ Nil(R).
RESULT 1.1.22. Let $R \in H_0$. Then every overring of R is in H_0.

PROOF. Let T be an overring of R. Then $T \in$ H, by Result 1.1.21. It remains to show that Nil(T) = Z(T). Obviously, Nil(T) ⊆ Z(T). Let $a/b \in Z(T)$ for some $a \in R$ and $b \in R \setminus Z(R)$. Then $(a/b)(c/d) = 0$ for some nonzero $c \in R$ and $d \in R \setminus Z(R)$. It follows that $ac = 0$ and hence $a \in Z(R)$ = Nil(R). Consequently, $a/b \in$ Nil(T). Thus, Nil(T) = Z(T).

A ring $R \in$ H is said to be φ-integrally closed if $\varphi(R)$ is integrally closed, see [4]. Griffin in [63] introduced the concept of Prüfer rings. A ring R is said to be a Prüfer ring if each finitely generated regular ideal of R is invertible, that is, if a is a finitely generated regular ideal of R, then $aa^{-1} = R$, where $a^{-1} = \{x \in \mathrm{tq}(R) \mid xa \subseteq R\}$. A ring $R \in$ H is said to be a φ-Prü fer ring if $\varphi(R)$ is a Prü fer ring, see [3]. The concept of valuation domains generalized in [55] to the context of arbitrary rings, where Froeschl III defined a ring R to be a chained ring if for $x, y \in R$, x divides y or y divides x in R. Motivated by this, Badawi defined the concept of φ-chained rings in [23]. A ring $R \in$ H is called a φ-chained ring if for each $x \in R_{\mathrm{Nil}(R)} \setminus \varphi(R)$, we have $x^{-1} \in \varphi(R)$. Recall from [65] that a domain $R$is called a pseudo-valuation domain (PVD) if, whenever a prime ideal p of R contains the product xy for any x, y in the quotient field of R, then $x \in p$ or $y \in p$. Every PVD R admits a canonically associated valuation overring V, in which every prime ideal of R is also a prime ideal of V and both R and V are local domains with the same maximal ideal, see [65, Theorem 2.7]. The study of PVD generalized to the context of arbitrary rings. A ring R is said to be a pseudo-valuation ring (PVR), if each prime ideal p of R is strongly prime, that is, if ap and bR are comparable for all $a, b \in R$. A ring $R \in$ H is said to be a φ-pseudo-valuation ring (φ-PVR). If each prime ideal p of R is φ-strongly prime, that is, if $xy \in \varphi(p)$ for $x, y \in R_{\mathrm{Nil}(R)}$, then $x \in \varphi(p)$ or $y \in \varphi(p)$. An integral domain R is called a Dedekind domain if every nonzero ideal of R is invertible, that is, if a is a nonzero ideal of R, then $aa^{-1} = R$, where $a^{-1} = \{x \in \mathrm{tq}(R) \mid xa \subseteq R\}$. Recall from [54] that an integral domain R is a Krull-domain if $R = \cap V_i$, where each V_i is a discrete valuation overring of R, and every nonzero element of R is a unit of all but finitely many V_i. An ideal a of a ring R is said to be a nonnil ideal if $a \not\subset$ Nil(R). A nonnil ideal a of a ring $R \in$ H is is said to be φ-invertible if $\varphi(a)$ is an invertible ideal of $\varphi(R)$, see [4]. If every nonnil ideal of $R \in$ H is φ-invertible, then R is said to be a φ-Dedekind ring, see [4]. A φ-chained ring R is said to be discrete if R has at most one nonnil prime ideal and every nonnil ideal of R is principal. Also, a ring $R \in$ H is said to be a φ-Krull ring if $\varphi(R) = \cap V_i$, where each V_i is a discrete φ-chained overring of $\varphi(R)$, andfor every non nilpotent element $x \in R$, $\varphi(x)$ is a unit of all but finitely many V_i, see [4].

Now, we list some definitions and results from the literature on ring theory and ringextensions which we used in this thesis.
RESULT 1.1.23. [Proposition 2.9] Let $R \in$ H. Then R is a φ-PVR if and only if R/Nil(R) is a PVD.
RESULT 1.1.24. [21, Corollary 7(3)] Every PVR is a φ-PVR.
RESULT 1.1.25. [3, Theorem 2.6] Let $R \in$ H. Then R is a φ-Prüfer ring if and only if R/Nil(R) is a Prüfer domain.
RESULT 1.1.26. [3, Theorem 2.14] Every φ-Prüfer ring is a Prüfer ring.
RESULT 1.1.27. [3, Theorem 2.17] Let $R \in$ H. Then R is a φ-Prüfer ring if and only if every

overring of $\varphi(R)$ is integrally closed.
RESULT 1.1.28. [3, Example 2.18] Let D be a Prüfer domain with Krull dimension n. Then $R = D(+)\mathrm{qf}(D)$ is a φ-Prüfer ring with Krull dimension n.
RESULT 1.1.29. [4, Theorem 2.5] Let $R \in \mathrm{H}$. Then R is a φ-Dedekind ring if and only if $R/\mathrm{Nil}(R)$ is a Dedekind domain.
RESULT 1.1.30. [4, Theorem 3.1] Let $R \in \mathrm{H}$. Then R is a φ-Krull ring if and only if $R/\mathrm{Nil}(R)$ is a Krull domain.
RESULT 1.1.31. [21, Proposition 3(3)] Let $R \in \mathrm{H}$ and let $x = a/b \in R_{\mathrm{Nil}(R)}$ for some $a \in R$ and for some $b \in R \setminus \mathrm{Nil}(R)$. Then $x \in \varphi(R)$ if and only if b divides a in R.
RESULT 1.1.32. [23, Proposition 2.2] A ring $R \in \mathrm{H}$ is a φ-chained ring if and only if forevery a, $b \in R \setminus \mathrm{Nil}(R)$, a divides b in R or b divides a in R.
RESULT 1.1.33. [23, Proposition 2.10] Every φ-chained ring is integrally closed and φ-integrally closed.
RESULT 1.1.34. [3, Theorem 2.7] Let $R \in \mathrm{H}$. Then R is a φ-chained ring if and only if $R/\mathrm{Nil}(R)$ is a valuation domain.
RESULT 1.1.35. [3, Corollary 2.8] Every φ-chained ring is a φ-Prüfer ring.
RESULT 1.1.36. [23, Proposition 2.9] Let $R \in \mathrm{H}$ be a φ-chained ring and let T be an a overring of R. Then T is a φ-chained ring and there exists a prime ideal p of R containing $\mathrm{Z}(R)$ such that $T = R_p$.
RESULT 1.1.37. [23, Lemma 3.1(1)] Let R and T be φ-chained rings with the same maximal ideal and the same total quotient ring. Then $R = T$.
DEFINITION 1.1.38. An integral domain R has property (#), if for any two distinct subsets Ω_1 and Ω_2 of $\mathrm{Max}(R)$, intersections $\cap_{m\in\Omega_1} R_m$ and $\cap_{m\in\Omega_2} R_m$ are distinct, see [62]. For example, every PID has property (#).
RESULT 1.1.39. [57, Proposition 4.9] Let R be a one-dimensional Prüfer domain with property (#). Consider the map Φ: {overrings of R} $\to$ {subsets of the set of valuation overrings of R} by $\Phi(T)$ = {valuation overrings of T} and the map Ψ: {subsets of the set of valuation overrings of R} $\to$ {overrings of R} by $\Psi(\{V_\alpha\}) = \cap V_\alpha$. Then Φ and Ψ are inverse maps and are both inclusion-reversing.
COROLLARY 1.1.40. *Let R be a one-dimensional Prüfer domain with property (#) and let Γ be the set of all valuation overrings of R. If $T = \cap_{V\in\Gamma\setminus\{V_1\}} V$, then $[R,T] = \{R,T\}$.*

PROOF. First, note that $R \subset T$ as $\Phi(T) \subset \Phi(R)$ by Result 1.1.39. Now, let $S \in [R,T]$. Since R is a one-dimensional Prüfer domain, $S = \cap_{V\in\Gamma} V$. Now, by Result 1.1.39, we have $\Phi(T) \subseteq \Phi(S) \subseteq \Phi(R)$. It follows that either $S = R$ or $S = T$.
COROLLARY 1.1.41. *Let R be a one-dimensional Prüfer domain with property (#) and let Γ be the set of all valuation overrings of R. If $T = \cap_{V\in\Gamma\setminus\{V_1,V_2\}} V$, then $[R,T] = \{R,S_1,S_2,T\}$, where $S_i = \cap_{V\in\Gamma\setminus\{V_i\}} V$ for $i = 1,2$. Moreover, S_1 and S_2 are not comparable.*
PROOF. First, note that S_1 and S_2 are distinct proper intermediate rings between R and T, by Result 1.1.39. Now, let $S \in [R,T] \setminus \{R,T\}$. Since R is a one-dimensional Prüfer domain, $S = \cap_{V\in\Gamma} V$. Now, by Result 1.1.39, we have $\Phi(T) \subset \Phi(S) \subset \Phi(R)$. It follows that either $S = S_1$ or $S = S_2$. Thus, $[R,T] = \{R,S_1,S_2,T\}$. Finally, if $S_1 \subset S_2$, then $\Phi(S_2) \subset \Phi(S_1)$, a contradiction. Similarly, $S_2 \not\subset S_1$.
DEFINITION 1.1.42. A ring extension $R \subseteq T$ is said to be a FIP-extension or $R \subseteq T$ sat- isfies FIP if $[R,T]$ is finite. For example, every minimal ring extension is a FIP extension. or the study of FIP property, see [1, 47, 69, 68, 43, 58, 44, 45, 13, 89, 88, 70, 7, 46].
DEFINITION 1.1.43. Consider the partially ordered set $([R,T], \subseteq)$. Then $R \subseteq T$ satisfies FCP if each

Industrial Engineering Journal
ISSN: 0970-2555
Volume : 52, Issue 7, July : 2023

chain in $[R, T]$ is finite, see [47]. For example, $F_2(X^2, Y^2) \subset F_2(X, Y)$ satisfies FCP. The length of $[R, T]$, denoted by $l\,[R, T]$, is the supremum of the lengthsof chains of R-subalgebras of T.

DEFINITION 1.1.44. Let R be an integral domain. If for each overring T of R, the canon- ical contraction map Spec(T) → Spec(R) is injective, then R is called an i-domain, see [92]. For example, Z is an i-domain.

The following result on i-domain is used frequently in this thesis:

RESULT 1.1.45. [92, Corollary 2.15] Let R be a domain. Then R is a local i-domain if and only if R' is a valuation domain.

DEFINITION 1.1.46. A proper ideal a of a ring R satisfying the condition that if $xyz \in a$ for $x, y, z \in R$, then $xy \in a$ or $yz \in a$ or $xz \in a$ is called a 2-absorbing ideal, see [27]. For example, $p \cap q$ is a 2-absorbing ideal for any prime ideals p, q of R.

DEFINITION 1.1.47. Let $R \subseteq T$ be a ring extension. If each element of T is a root of some polynomial in $R[X]$, then $R \subseteq T$ is called an algebraic extension. Moreover, if at least one of the coefficients of that polynomial is a unit of R, then $R \subseteq T$ is called a P-extension, see [60]. For example, $Z \subset Z[1/2]$ is a P-extension.

RESULT 1.1.48. [29, Theorem 2.2] Let $R \in H_0$. Then $R \subseteq \mathrm{tq}(R)$ is a P-extension if and only if R' is a Prüfer ring.

RESULT 1.1.49. [47, Lemma 3.8] Let $R \subset T$ be a ring extension such that R is integrally closed in T, let $u \in T$ and $p \in$ Spec(R). Suppose that u is a root of some polynomialin $R[X]$ that has at least one coefficient in $R \setminus p$. Then u satisfies at least one of the following two conditions:

(i) $u/1 \in R_p$;

(ii) $u/1$ is a unit of T_p and $(u/1)^{-1} \in R_p$.

RESULT 1.1.50. [47, Lemma 3.9] Let $R \subset T$ be a P-extension such that R is integrally closed in T. If $q \in$ Spec(T) and $p = q \cap R$, then $R_p = T_p$.

DEFINITION 1.1.51. Let $R \subseteq T$ be a ring extension. If every intermediate ring is integrally closed in T, then the pair (R, T) is called a normal pair, see [35]. For example, (Z, Q) is a normal pair.

RESULT 1.1.52. [51, Proposition 3.1] Let $R \subseteq T$ be a ring extension. Then the following are equivalent:

(i) (R, T) is a normal pair.

(ii) (R_p, T_p) is a normal pair for all $p \in$ Spec(R).

(iii) (R_m, T_m) is a normal pair for all $m \in$ Max(R).

RESULT 1.1.53. [47, Theorem 6.8] Let (R, m) be a local ring and T be a ring containing R. Then the pair (R, T) is normal if and only if there exists $q \in$ Spec(R) such that $T = R_q$, $q = qT$ and R/q is a valuation domain. Under these conditions, T/q is necessarily the quotient field of R/q.

DEFINITION 1.1.54. Let $R \subseteq T$ be a ring extension. If for any prime ideal q of T and $p = q \cap R$, the ring T/q is algebraic over R/p, then $R \subseteq T$ is called a residually algebraic extension. Moreover, the pair (R, T) is called a residually algebraic pair if for any ring S in $[R, T]$, the extension $R \subseteq S$ is residually algebraic, see [42] and [11]. For example, (Z, Q) is a residually algebraic pair.

RESULT 1.1.55. [11, lemma 2.9] Let (R, T) be a residually algebraic pair of integral domains such that R is integrally closed in T. Then the following hold:

(i) If $q \in$ Spec(T), then $R_q \cap R = T_q$.

(ii) Spec(T) = $\{pT : pT \neq T, p \in \mathrm{Spec}(R)\}$.

DEFINITION 1.1.56. A ring R is said to be a quasi-valuation ring if x divides y or y divides x in R for all $x, y \in R \setminus Z(R)$, see [55]. For example, tq(R) is a quasi-valuation ring for every ring R.

DEFINITION 1.1.57. Let $R \subset T$ be a ring extension. Then $R \subset T$ satisfies INC if for any two distinct prime ideals q_1, q_2 of T such that $q_1 \cap R = q_2 \cap R$, we have q_1, q_2 are incomparable, see [74].

Moreover, the pair (R, T) is said to be an INC-pair if $R \subset S$ satisfies INC for all $S \in [R, T] \setminus \{R\}$, see [74]. For example, $(K, K \times K)$ is an INC-pair for any field K. Here, K is a subring of $K \times K$ via the diagonal map $r \rightarrow (r, r)$, for all $r \in K$.

RESULT 1.1.58. [38, Corollary 4] An extension $R \subseteq T$ is a P-extension if and only if (R, T) is an INC-pair.

RESULT 1.1.59. [32, Theorem 3.1] Let V be a valuation domain with maximal ideal m such that $V = F + m$, where F is a field contained in V. Let D be a proper subring of F and set $R = D + m$. Let T be a ring containing R. Then T is an overring of R if and only if T is an overring of V or $T = E + m$ for some subring E of F containing D.

RESULT 1.1.60. Let $R \subset T$ be a ring extension such that for each $t \in T \setminus R$, $t^{-1} \in R$. Then R is integrally closed in T.

PROOF. Let $t \in T \setminus R$ be integral over R. Then $t^n + r_{n-1}t^{n-1} + \cdots + r_1 t + r_0 = 0$ for some $r_0, r_1, \ldots, r_n \in R$. It follows that $t = -(t^{1-n})(\sum_{i=0}^{n-1} r_i t^i) \in R$. Thus, R is integrally closed in T.

RESULT 1.1.61. [52, Theorem] Let $R \subseteq T$ be an integrally closed extension such that there exist $t_1, t_2, \ldots, t_n$ in T such that T is integral over $R[t_1, t_2, \ldots, t_n]$. If a prime ideal p of T is maximal and minimal with respect to prime ideals of T whose intersection with$R$ is $R \cap p$, then there exists an $s \in R \setminus (R \cap p)$ such that $T_s = R_s$.

RESULT 1.1.62. [5, Theorem 5.10] Let $R \subseteq T$ be an integral extension and p be a prime ideal of R. Then there exists a prime ideal q of T such that $p = q \cap R$.

RESULT 1.1.63. [5, Corollary 5.8] Let $R \subseteq T$ be an integral extension and m be a maxi- mal ideal of T. Then $m \cap R$ is a maximal ideal of R.

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