Industrial Engineering Journal

ISSN: 0970-2555

Volume: 54, Issue 1, January:2025

EXISTENCE OF SOLUTION OF DYNAMIC APPROACH MODEL OF MODIFIED RENEWAL EQUATION

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ABSTRACT:

This article establishes the existence and uniqueness of the solutions for a general utility function. This paper explains the model of dynamic structure of the renewal equation arising in optimization problem. Further it establishes a functional method proved it by using dilation principle and contraction principle.The present work derives the convergent of dynamic model of renewal equation to an unique solution.

KEYWORDS: Dynamic programming, multistage allocation, fixed point, renewal equation, contraction principle,dilation principle.

INTRODUCTION:

Dynamic programming is a mathematical technique dealing with the optimization of multistage decision process which is based on “Bellmann’s principle of optimality”. As a result of this process some functional equation arise as a certain type of relationship between stage transformation and return function involving state and decision variables. Bhakta and Mitra [6], Bellmann have established a number of theorems for the existence and uniqueness of the solution of the functional equation arising in dynamic programming. I. Dmitry et al [2] studied a multi structural framework with dynamic considerations. We have established the existence and uniqueness of the solutions for a utility function. Bhardwaj et al. [4] summarized contractive mapping of different types and discussed on their fixed-point theorems. He considered many types of mappings and analyzed the relationship amongst them. In our work we have established existence and uniqueness of the solution of the renewal equation arising in dynamic programming. We have proved it in a different method using dilation principle. In the present model we have considered a dynamic model of renewal equation.

A multistage allocation process:

Let us describe a multistage allocation process and assume a positive quantity x, which divide into two parts y and $x-y$, obtaining from first quantity a return g (x,y) and from second $T(x,y)$. To maximize the total return in first stage, determine the maximum of the function $g(x,y)+T(x,y)$, for all y in $[0,x]$.

If we set $T(f,y)=p(y)+q(x-y)+f\big(ay+b(x-y)\big)$

$$and\ f(x,y)=\max_{0\le y\le x} T(f,y)=\max_{0\le y\le x}\ [g(x,y)+T(x,y)]\quad ,$$

Now consider a multistage allocation process.

Suppose after obtaining the returns g(x,y) and $T(x,y)$ i.e. $f_1(x,y)$, the original quantity y and $x-y$ reduces to ay and $b(x-y)$ respectively, where $0\le a<1$, $0\le b<1$.

The remaining quantity is $ay+b(x-y)=y_1+(x_1-y_1)$, for $0\le y_1\le x_1$.

As a result of this new allocation, we obtain the return $g_1(x_1,y_1)+\ T_1(f_1,y_1)$ in second stage.

We set $G(x,y,T_1(f,y))=\ g_1(x_1,y_1)+\ T_1(f_1,y_1)$.

The total return for the two stage process is $g(x,y)+\ G(x,y,T_1(f,y))$.

The maximum return is obtained by maximizing the above function. Let us set

$$f_2(x,y) = \max_{0\le y\le x}[\ g(x,y) + \ G(x,y,T_1(f,y))]$$

Then $$f_N(x,y) = \max_{0\le y\le x}[g(x,y) + G(x,y,T_{N-1}(f,y))]$$

If N is large, letting N→ ∞. Hence in place of a sequence of equations as discussed above have a single equation.

$$f(x,y) = \sup_{0\le y\le x}[g(x,y) + G(x,y,T(f,y)] \qquad \cdots (1)$$

Assume that S is the state space and D is the decision space. In this chapter, consider X, Y to be Banach spaces and S $\subseteq X$, D$\subseteq Y$. Let B(S) denote the metric space of all real valued bounded functions on S.

$d(\Psi_1,\Psi_2) = \operatorname{Sup}_{x\in S}|\Psi_1(x) - \Psi_2(x)|$,for $\Psi_1,\ \Psi_2$ in B(S).

Then (B(S), d) is a complete metric space.

To prove existence theorems, it is essential to state the following two lemmas.

Lemma1: Let (S,d) be a complete metric space and let A be a mapping of S into itself satisfying the following conditions.

(i) For any x,y in S, $d(Ax,Ay) \le \phi\big(d(x,y)\big)$.

Where $\phi : [0,\infty) \to [0,\infty)$ is non decreasing continuous on the right and $\phi(r) < r$ for $r > 0$.

(ii) For every x in S, there is a positive number λ_x such that $d(x,A^n x) \le \ \lambda_x$, for all n.

Then A has a unique fixed point.

Lemma 2: Let (S,d) be a complete metric space and let A be a mapping of S into itself satisfying $d(Ax,Ay) \le \phi\big(d(x,y)\big)$,

for all x,y in S.

Where $\phi : [0,\infty) \to [0,\infty)$ is non decreasing and for every positive r, the series $\sum \phi^n(r)$ is convergent .

Then A has a unique fixed point.

Existence theorems:

Theorem 1: Suppose the following conditions hold.

(i) g and G are bounded functions.

(ii) $|G(x,y,z_1) - G(x,y,z_2)| \ge \lambda(\ |z_1 - z_2|)$.

For all (x,y,z_1) and (x,y,z_2) in S$\times D \times R$ and $\lambda < 1$.

Then the functional equation (1) possesses a unique bounded solution on S.

Proof: Let us define a mapping A on B(S) by

A h = Ψ for h $\in B(S)$, $\Psi(x) = \inf_{0\le y\le x}[g(x,y) + G(x,y,h(f,y))]$, for $x \in S$.

For i = 1,2 and $x \in S$. $\Psi_i(x) = \inf_{0\le y\le x}[g(x,y) + G(x,y,h_i(f,y))]$.

Let ϵ be any positive number .Then we can choose points y_1,y_2 in D such that

$$\Psi_1(x) < \big[\,g(x,y_1) + \ G\big(x,y_1,h_1(f,y_1)\big)\big] - \epsilon \qquad \cdots (3)$$

$$\Psi_2(x) > [g(x,y_2) + G(x,y_2,h_2(f,y_2))] - \epsilon \qquad \cdots (4)$$

$$\Psi_1(x) - \Psi_2(x) < \ \lambda\, d(h_1,h_2) + \epsilon \qquad \dots (5)$$

$$\text{Again } \Psi_1(x) - \Psi_2(x) > \ -\lambda\, d(h_1,h_2) - \epsilon \qquad \cdots (6)$$

Hence we have $|\Psi_1(x) - \Psi_2(x)| \le \lambda\, d(h_1,h_2)$.

Let a function $\phi: [0,\infty) \to [0,\infty)$ be defined by $\phi(r) = r\lambda$, is non decreasing, continuous and $\phi(r) < r$, for all $r > 0$.

Then $d(Ah_1,Ah_2) \le \lambda\, d(h_1,h_2)$. Thus A is a contraction mapping by lemma1.

Hence the equation (1) possesses a solution if either A or A^{-1} has a fixed point.

We have $d(A^{-1}h_1, A^{-1}h_2) \le \lambda\, d(h_1,h_2) = \phi\big(d(h_1,h_2)\big)$, where $0 < \lambda < 1$.

Again if we set $h_n = A^{-n}h$, for $h \in B(S)$ and n =1, 2, 3........

Then $h_n(x) = \Psi_{n-1}(x) = \inf\big[g(x,y) + G\,\big(x,y,h_{n-1}(f,y)\big)\big]$

$| \quad h(x) - h_n(x) \,| \le |h(x)| + \inf\limits_{y\in D}|g(x,y)| + \inf\limits_{y\in D}\big|G\big(x,y,h_{n-1}(f,y)\big)\big| \le k_1 + k_2 + k_3 = \lambda_k\ (say)$.

Then $d\,(\,h\,, A^{-n}h) \le \ \lambda_k$. For n = 1, 2… by lemma1.

Thus the mapping A possesses a unique fixed point. Thus the functional equation (1) has a unique bounded solution.

Theorem 2: Suppose that $|G\,(x,y,z_1) - G(x,y,z_2)| \ge \ \phi|z_1(x) - z_2(x)|$

Where $\phi : [0,\infty) \to [0,\infty)$ is a non decreasing positive continuous function on the right and $\phi(r) < r$ for $r > 0$. Then the functional equation (1) possesses a unique solution on S.

Proof: Let $\phi : [0,\infty) \to [0,\infty)$ be defined by $\phi(r) = \frac{r}{\lambda}, r > 0, \lambda > 1$, is non decreasing and continuous .

Then $\sum \phi^n(r) = r\sum \frac{1}{\lambda^n}$ is convergent for every positive number r.

Again proceeding as in theorem (1) we have $d(A^{-1}h_1, A^{-1}h_2) \le \phi\big(d(h_1,h_2)\big)$.

Hence by lemma 2 the functional equation (1) possesses a unique bounded solution on S.

Theorem 3: Under the following conditions the functional equation (1) possesses a unique solution on S.

(i) G is bounded.

(ii) $|G(x,y,z_1) - G(x,y,z_2)| \le \ \lambda(|z_1 - z_2|)$ for all (x,y,z_1) and (x,y,z_2) in $S \times D \times R$ and $\lambda < 1$.

(iii) Let $\{y_n\}$ be the arbitrary sequence in D and $\epsilon_n = d(A^{-1}y_n, y_{n+1}), n = 0,1,2, \dots\dots$

Proof: Let us define a mapping A on B(S) by $Ah = \Psi$ for $h \in B(S)$, Where $\Psi(x) = \inf\limits_{0\le y\le x}\big[g(x,y) + G\big(x,y,h(f,y)\big)\big]$ Proceeding as in theorem (1) and (2), using dilation principle we obtain $d(A^{-1}h_1, A^{-1}h_2) \le \frac{1}{\lambda}\, d(h_1,h_2)$, 0< $\frac{1}{\lambda} < 1$,

Let $\{x_n\}$ be a sequence in B(S) defined by $x_n = A^{-1}\ x_{n-1}, n = 1,2, \dots\dots$

Then $d(x_m, x_n) \le \frac{1}{\lambda}\ d\ (x_{m-1}, x_{n-1})$,for 1≤ $m \le n$.

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.$$

$$\le \frac{1}{\lambda^m}\left[d(x_0,x_1) + \frac{1}{\lambda}d(x_0,x_1) + \frac{1}{\lambda^2}d(x_0,x_1) + \cdots\cdots + \frac{1}{\lambda^{n-m-1}}d(x_0,x_1)\right]$$

$$\le \frac{1}{\lambda^m(1-\frac{1}{\lambda})}d(x_0,x_1)$$

Since B(S) is a complete metric space, $\{x_n\}$ converges to a fixed point x^*(say) of A in S i.e., x^* is a solution of the functional equation (1).

CONCLUSION:

The existence and uniqueness of the solutions for a general utility function obtained. The convergent of dynamic model of renewal equation to an unique solution has been derived.

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