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DERIVATIONS AND TRANSLATIONS OF NEUTROSOPHIC FUZZY POSITIVE IMPLICATIVE IDEALS OF BCK-ALGEBRA

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ABSTRACT
This study explores the application of the Derivations and Translations concept to various Neutrosophic Fuzzy structures, including NFSA, NFI, NFII, and NFPII. By introducing the notions of DNFSA, DNFI, DNFII, and DNFPII, we uncover distinct results, examine the interconnections between these structures, furthermore we introduced Neutrosophic fuzzy translation to Neutrosophic fuzzy positive implicative ideals in BCK-algebras and investigate their related properties.

Keywords:
BCK-algebras, Derivations, DNFSA, DNFI, DNFII, and DNFPII, Fuzzy translation, Neutrosophic fuzzy translation , Neutrosophic fuzzy implicative-ideal, Neutrosophic fuzzy positive implicative ideal.

I. Introduction

The concept of BCK-algebras was introduced by K. Iseki and Y. Imai in 1966 [2], pioneering a wave of research into their various properties. Later, Iseki and Tanaka [3] introduced sub-algebras, ideals (PII's) in BCK-algebras. Zadeh [12] pioneered the concept of fuzzy sets in 1965 as a means of representing uncertainty in the real world. Xi [11] defined fuzzy ideals and fuzzy implicative ideals in 1991, delving into the study of fuzzy BCK-algebras. Building on Atanassov's [1] work, Jun and Kim [4] investigated intuitionistic fuzzy sub-algebras and ideals in BCK-algebras. This section defines and exemplifies intuitionistic fuzzy sub-algebras (IFSA) and intuitionistic fuzzy ideals (IFI) in BCK-algebras, along with related results. Meng [6] established implicative ideals in BCK-algebras, while fuzzy implicative ideals (FII) were introduced and their properties explored by Meng et.al. [7]. In [9] Satyanarayana et. al., develop the concept of derivations of intuitionistic fuzzy positive implicative ideals of BCK-algebra. Satyanarayana and Durga Prasad [10] then introduced on fuzzy ideals in BCK-algebras and examined their properties. Lee et al. [5] in (2009) examined fuzzy translations in fuzzy subalgebras and beliefs in BCK/BCI-algebras. Exploring the connections between fuzzy translations, extensions and multiplications. In [8] Satyanarayana et. al., introduced intuitionistic fuzzy translations of implicative ideals of BCK-algebras, now we are generalized [8, 9] work into Neutrosophic fuzzy logic. This historical overview showcases the significant contributions to BCK-algebra development, paving the way for future research.
This paper explores the application of Left-Right Derivation ((L, R)-D) and Right-Left Derivation ((R, L)-D) a particular derivative approach to develop a deeper understanding (NFSA, NFI, and DNFI). We introduce four new concepts: Derivations of Neutrosophic fuzzy sub-algebra (DNFSA), Derivations of Neutrosophic fuzzy ideal (DNFI), Derivations of Neutrosophic fuzzy implicative ideal (DNFII), and Derivations of Neutrosophic fuzzy positive implicative ideal (DNFPII). Our objective is to explore the interrelationships between these concepts, uncover specific outcomes, and investigate various associated properties, ultimately testing a range of related residency outcomes. Finally, we discussed Neutrosophic fuzzy translation to Neutrosophic fuzzy positive implicative ideals in BCK-algebras, analyzing some of their properties.

The following abbreviations are utilized throughout this paper:

- Ɠ denotes BCK-Algebra
- NFS : Neutrosophic fuzzy set
- NFPII : Neutrosophic fuzzy positive implicative ideal
- DNFCI : Derivations of Neutrosophic fuzzy commutative ideal
- LDI : Left Derivation Ideal
- RDI : Right Derivation Ideal
- RDNFI: Right derivation Neutrosophic fuzzy ideal
- LDNFI: Left derivation Neutrosophic fuzzy ideal
- RDNFII : Right derivation Neutrosophic fuzzy implicative ideal
- LDNFII : Left derivation Neutrosophic fuzzy implicative ideal
- $\mathcal{NFI}$: Neutrosophic Fuzzy Ideal
- $\mathcal{NFPII}$: Neutrosophic Fuzzy positive implicative ideal
- NFT : Neutrosophic fuzzy Translation
- $\mathrm{NF}^{\beta} - \mathrm{T}$: Neutrosophic fuzzy β- translation

II. Preliminaries

Definition 2.1: Ɠ comprises (≠φ) set equipped with a binary operation ⁎ and a constant 0, if it satisfies the following axioms ∀ ƀ, ɰ, ɣ ∈ Ɠ.

BCK-1) $\big((ƀ ⁎ ɰ) ⁎ (ƀ ⁎ ɣ)\big) ⁎ (ɣ ⁎ ɰ) = 0$

BCK-2) $\big(ƀ ⁎ (ƀ ⁎ ɰ)\big) ⁎ ɰ = 0$

BCK-3) $ƀ ⁎ ƀ = 0$

BCK-4) $0 ⁎ ƀ = 0$

BCK-5) $ƀ ⁎ ɰ = 0$ and $ɰ ⁎ ƀ = 0$ implies $ƀ = ɰ$.

Define a binary relation ≤ on Ɠ by $ƀ \leq ɰ \Leftrightarrow ƀ ⁎ ɰ = 0$. This yields a partial order on (Ɠ, ≤) with minimal element 0. Furthermore, (Ɠ,⁎, 0) constitutes a Ɠ iff It adheres to the following rules

i) $\big((ƀ ⁎ ɰ) ⁎ (ƀ ⁎ ɣ)\big) \leq (ɣ ⁎ ɰ)$

ii) $\big(ƀ ⁎ (ƀ ⁎ ɰ)\big) \leq ɰ$

iii) $ƀ \leq ƀ$

iv) $0 \leq ƀ$

v) $ƀ \leq ɰ$ and $ɰ \leq ƀ$ implies $ƀ = ɰ$, ∀ ƀ, ɰ, ɣ ∈ Ɠ.

Ɠ is distinguished by the following attributes:

(P-1) $ƀ ⁎ 0 = ƀ$

(P-2) $ƀ ⁎ ɰ \leq ƀ$

(P-3) $(ƀ ⁎ ɰ) ⁎ ɣ = (ƀ ⁎ ɣ) ⁎ ɰ$

(P-4) $(ƀ ⁎ ɣ) ⁎ (ɰ ⁎ ɣ) \leq (ƀ ⁎ ɰ)$

(P-5) $ƀ ⁎ \big(ƀ ⁎ (ƀ ⁎ ɰ)\big) = ƀ ⁎ ɰ$

(P-6) $ƀ \leq ɰ \Rightarrow ƀ ⁎ ɣ \leq ɰ ⁎ ɣ$ and $ɣ ⁎ ɰ \leq ɣ ⁎ ƀ$

(P-7) $ƀ ⁎ ɰ \leq ɣ \Rightarrow ƀ ⁎ ɣ \leq ɰ$ ∀ ƀ, ɰ, ɣ ∈ Ɠ.

An ideal of Ɠ if (i-1) 0 ∈ 𝔜, (i-2) ƀ ⁎ ɰ and ɰ ∈ 𝔜 implies ƀ ∈ 𝔜 ∀ ƀ, ɰ ∈ Ɠ.

Ɠ is called implicative if $ƀ = ƀ ⁎ (ɰ ⁎ ƀ)$, ∀ ƀ, ɰ ∈ Ɠ.

Ɠ is considered to be positive implicative when $(ƀ ⁎ ɰ) ⁎ ɣ = (ƀ ⁎ ɣ) ⁎ (ɰ ⁎ ɣ)$, ∀ ƀ, ɰ, ɣ ∈ Ɠ.

A non-empty subset 𝔜 of Ɠ is a sub-algebra of Ɠ if the binary operation applied to any elements ƀ and ɰ in 𝔜 yields a result within 𝔜.

A subset 𝔜 of Ɠ is an ideal of Ɠ if it meets the following criteria: **(I-1)** it contains the additive identity (0 ∈ 𝔜), and **(I-2)** for any elements ƀ and ɰ in Ɠ, if ƀ ⁎ ɰ is in 𝔜 and ɰ is in 𝔜, then ƀ is also in 𝔜.

An implicative ideal(II) if **(II-1)** and **(II-3)** $(ѣ \ast (ɰ \ast ѣ)) \ast ɣ \in 𝔜$ and $ɣ \in 𝔜$ imply $ѣ \in 𝔜$ $\forall$ $ѣ, ɰ, ɣ \in Ɠ$.

Let's briefly review the concepts of Fuzzy sets(FS's) and Intuitionistic fuzzy set(IFS's) before proceeding.

A FS in $Ɠ$ is a function $\mathbb{P}: Ɠ \to [0, 1]$ and the complement of $\mathbb{P}$ denoted by $\overline{\mathbb{P}}$ the $𝔜$ on $Ɠ$ given by $\overline{\mathbb{P}}(ѣ) = 1 - \mathbb{P}(ѣ)$ $\forall$ $ѣ \in Ɠ$.

Consider FS's $\mathbb{P}$ and $Ч$ defined on $Ɠ$. For any membership values m and n in the unit interval $[0, 1]$, we define the upper m -level cut of $\mathbb{P}$ as $U(\mathbb{P}, m) = \{ѣ \in Ɠ |\ \mathbb{P}_{Ⱥ}(ѣ) \geq m\}$ and the lower n -level cut of $Ч$ as $L(Ч, n) = \{ѣ \in Ɠ | Ч(ѣ) \leq n\}$. These level cuts can be used to characterize the properties of FS's $\mathbb{P}$ and $Ч$.

An IFS $Ⱥ$ in a ($\neq \varphi$) set $Ɠ$ can be represented as: $Ⱥ = \{ѣ, \mathbb{P}_{Ⱥ}(ѣ), Ч_{Ⱥ}(ѣ) | ѣ \in Ɠ\}$ where the functions $\mathbb{P}_{Ⱥ}: Ɠ \to [0, 1]$, and $Ч_{Ⱥ}: Ɠ \to [0, 1]$ denoted the degree of membership and non-membership of each element $ѣ \in Ɠ$ to the set $Ⱥ$ respectively and $0 \leq \mathbb{P}_{Ⱥ}(ѣ) + Ч_{Ⱥ}(ѣ) \leq 1$ $\forall$ $ѣ \in Ɠ$.

Let $\mathbb{P}, ɗ$ and $Ч$ be the FS's on $Ɠ$. For $m, k, n \in [0, 1]$ the set $U(\mathbb{P}, m) = \{ѣ \in Ɠ |\ \mathbb{P}_{Ⱥ}(ѣ) \geq m\}$, $U(ɗ, k) = \{ѣ \in Ɠ |\ ɗ_{Ⱥ}(ѣ) \geq k\}$ are called upper m-level, upper k-level cuts of $\mathbb{P}$ and $ɗ$ and the set $L(Ч, n) = \{ѣ \in Ɠ | Ч_{Ⱥ}(ѣ) \leq n\}$ is called lower n-level cut of $Ч$ and can be used to characterize of $\mathbb{P}, ɗ$ and $Ч$.

A NFS $Ⱥ$ in a non-empty set $Ɠ$ is an object having the form $Ⱥ = \{ѣ, \mathbb{P}_{Ⱥ}(ѣ), ɗ_{Ⱥ}(ѣ), Ч_{Ⱥ}(ѣ) | ѣ \in Ɠ\}$ where the functions $\mathbb{P}_{Ⱥ}: Ɠ \to [0, 1]$, $ɗ_{Ⱥ}: Ɠ \to [0, 1]$ and $Ч_{Ⱥ}: Ɠ \to [0, 1]$ denoted the degree of membership, indeterminacy and non-membership of each element $ѣ \in Ɠ$ to the set $Ⱥ$ respectively and $0 \leq \mathbb{P}_{Ⱥ}(ѣ) + ɗ_{Ⱥ}(ѣ) + Ч_{Ⱥ}(ѣ) \leq 1$ $\forall$ $ѣ \in Ɠ$.

Let $Ɠ$ stand for a BCK-algebra.

A map $\Lambda: Ɠ \to Ɠ$ is referred to as (L, R)-D of $Ɠ$ if:

$\Lambda(ѣ \ast ɰ) = (\Lambda(ѣ) \ast ɰ) \wedge (ѣ \ast \Lambda(ɰ))$, $\forall$ $ѣ, ɰ \in Ɠ$.

A map $\Lambda: Ɠ \to Ɠ$ is referred to as (R, L)-D of $Ɠ$ if:

$\Lambda(ѣ \ast ɰ) = (ѣ \ast \Lambda(ɰ)) \wedge (\Lambda(ѣ) \ast ɰ)$, $\forall$ $ѣ, ɰ \in Ɠ$.

A mapping $\Lambda: Ɠ \to Ɠ$ is defined as a derivation of $Ɠ$ if it simultaneously satisfies (L, R)-D and (R, L)-D conditions on $Ɠ$.

Suppose $(Ɠ, \ast, 0)$ is a $Ɠ$, $\Lambda: Ɠ \to Ɠ$ is a self-map, and $Ⱥ$ is a non-empty subset of $Ɠ$ and $ѣ, ɰ, ɣ \in Ɠ$ is called **(i)** LDI of the $Ɠ$ if it complies with: **(D-1)** $0 \in Ⱥ$ and **(LD-2)** $\Lambda(ѣ) \ast ɰ \in Ⱥ$ and $\Lambda(ɰ) \in 𝔜$ entail that $\Lambda(ѣ) \in 𝔜$, $\forall$ $ѣ, ɰ \in Ɠ$.

(ii) RDI of the $Ɠ$ if it complies with:

(D-1) $0 \in Ⱥ$ and **(RD-2)** $ѣ \ast \Lambda(ɰ) \in Ⱥ$ and $\Lambda(ɰ) \in 𝔜$ entail that $\Lambda(ѣ) \in 𝔜$, $\forall$ $ѣ, ɰ \in Ɠ$.

And is designated as a derivation ideal(DI) of $Ɠ$, **(D-1)** and **(D-2)** $\Lambda(ѣ \ast ɰ) \in Ⱥ$ and $\Lambda(ɰ) \in 𝔜$ entail that $\Lambda(ѣ) \in 𝔜$, $\forall$ $ѣ, ɰ \in Ɠ$.

A subset $Ⱥ$ of $Ɠ$, non-empty and containing $ѣ, ɰ, ɣ \in Ɠ$, is called a derivation implicative ideal (DII) of $Ɠ$ if it meets specific criteria:

(DII-1) $0 \in Ⱥ$

(DII-2) $\Lambda((ѣ \ast (ɰ \ast ѣ)) \ast ɣ) \in Ⱥ$ and $\Lambda(ɣ) \in Ⱥ$ entail that $\Lambda(ѣ) \in Ⱥ$. The analogous concept holds for left and right DII's.

Definition 2.2: A self-mapping of a $Ɠ$ is termed as regular if it meets the criterion $\Lambda(0) = 0$.

Corollary 2.3 : $Ɠ$ possesses a regular derivation.

Proposition 2. 4: For all $ѣ, ɰ$ in $Ɠ$, the following properties hold, given that Λ is a regular derivation on $Ɠ$.

(i) $\Lambda(ƀ) \leq ƀ$
(ii) $\Lambda(ƀ) * ɰ \leq ƀ * \Lambda(ɰ)$
(iii) $\Lambda(ƀ * ɰ) = \Lambda(ƀ) * ɰ \leq \Lambda(ƀ) * \Lambda(ɰ)$
(iv) $\Lambda^{-1}(0) = \{ƀ \in Ɠ \mid \Lambda(ƀ) = 0\}$ is a sub algebra of Ɠ and $\Lambda^{-1}(0) \subset Ɠ$.

Definition 2.5: A fuzzy set $\mathbb{P}$ in Ɠ qualifies as a fuzzy positive implicative ideal if it adheres to
$(\mathcal{FPII} - 1)$ $\mathbb{P}(0) \geq \mathbb{P}(ƀ)$
$(\mathcal{FPII} - 2)$ $\mathbb{P}(ƀ * ɣ) \geq min\{\mathbb{P}((ƀ * ɰ) * ɣ), \mathbb{P}(ɰ * ɣ)\}$, $\forall$ ƀ, ɰ $\in$ Ɠ.

Definition 2.6: A NFS $Ⱥ = (\mathbb{P}_{Ⱥ}, ȡ_{Ⱥ}, Ч_{Ⱥ})$ in Ɠ is called $\mathcal{NFI}$ (Neutrosophic fuzzy ideal) of Ɠ if it satisfies:
$(\mathcal{NFI} - 1)$ $\mathbb{P}_{Ⱥ}(0) \geq \mathbb{P}_{Ⱥ}(ƀ)$, $ȡ_{Ⱥ}(0) \geq ȡ_{Ⱥ}(ƀ)$ and $Ч_{Ⱥ}(0) \leq Ч_{Ⱥ}(ƀ)$
$(\mathcal{NFI} - 2)$ $\mathbb{P}_{Ⱥ}(ƀ) \geq min\{\mathbb{P}_{Ⱥ}(ƀ * ɰ), \mathbb{P}_{Ⱥ}(ɰ)\}$
$(\mathcal{NFI} - 3)$ $ȡ_{Ⱥ}(ƀ) \geq min\,\{ȡ_{Ⱥ}(ƀ * ɰ), ȡ_{Ⱥ}(ɰ)\}$
$(\mathcal{NFI} - 4)$ $Ч_{Ⱥ}(ƀ) \leq max\,\{Ч_{Ⱥ}(ƀ * ɰ), Ч_{Ⱥ}(ɰ)\}$ $\forall$ ƀ, ɰ $\in$ Ɠ.

Definition 2.7: A NFS $Ⱥ = (\mathbb{P}_{Ⱥ}, ȡ_{Ⱥ}, Ч_{Ⱥ})$ in Ɠ is called $\mathcal{NFII}$ (Neutrosophic fuzzy implicative ideal) of Ɠ if it satisfies:
$(\mathcal{NFII} - 1)$ $\mathbb{P}_{Ⱥ}(0) \geq \mathbb{P}_{Ⱥ}(ƀ)$, $ȡ_{Ⱥ}(0) \geq ȡ_{Ⱥ}(ƀ)$ and $Ч_{Ⱥ}(0) \leq Ч_{Ⱥ}(ƀ)$
$(\mathcal{NFII} - 2)$ $\mathbb{P}_{Ⱥ}(ƀ) \geq min\left\{\mathbb{P}_{Ⱥ}\left(\left(ƀ * (ɰ * ƀ)\right) * ɣ\right), \mathbb{P}_{Ⱥ}(ɣ)\right\}$
$(\mathcal{NFII} - 3)$ $ȡ_{Ⱥ}(ƀ) \geq min\left\{ȡ_{Ⱥ}\left(\left(ƀ * (ɰ * ƀ)\right) * ɣ\right), ȡ_{Ⱥ}(ɣ)\right\}$
$(\mathcal{NFII} - 4)$ $Ч_{Ⱥ}(ƀ) \leq max\left\{Ч_{Ⱥ}\left(\left(ƀ * (ɰ * ƀ)\right) * ɣ\right), Ч_{Ⱥ}(ɣ)\right\}$ $\forall$ ƀ, ɰ, ɣ $\in$ Ɠ.

Definition 2.8: A NFS $Ⱥ = (\mathbb{P}_{Ⱥ}, ȡ_{Ⱥ}, Ч_{Ⱥ})$ in Ɠ is called $\mathcal{NFPII}$ (Neutrosophic fuzzy positive implicative ideal) of Ɠ if it satisfies:
$(\mathcal{NFPII} - 1)$ $\mathbb{P}_{Ⱥ}(0) \geq \mathbb{P}_{Ⱥ}(ƀ)$, $ȡ_{Ⱥ}(0) \geq ȡ_{Ⱥ}(ƀ)$ and $Ч_{Ⱥ}(0) \leq Ч_{Ⱥ}(ƀ)$
$(\mathcal{NFPII} - 2)$ $\mathbb{P}_{Ⱥ}(ƀ * ɣ) \geq min\{\mathbb{P}_{Ⱥ}((ƀ * ɰ) * ɣ), \mathbb{P}_{Ⱥ}(ɰ * ɣ)\}$
$(\mathcal{NFPII} - 3)$ $ȡ_{Ⱥ}(ƀ * ɣ) \geq min\{ȡ_{Ⱥ}((ƀ * ɰ) * ɣ), ȡ_{Ⱥ}(ɰ * ɣ)\}$
$(\mathcal{NFPII} - 4)$ $Ч_{Ⱥ}(ƀ * ɣ) \leq max\{Ч_{Ⱥ}((ƀ * ɰ) * ɣ), Ч_{Ⱥ}(ɰ * ɣ)\}$ $\forall$ ƀ, ɰ, ɣ $\in$ Ɠ.

Example 2.9 : Let $Ɠ = \{0, ŧ_0, ɯ_0, ɒ_0\}$ be a $\mathcal{BCK}$ -algebra with the given table.

$*$	0	$ŧ_0$	$ɯ_0$	$ɒ_0$
0	0	0	0	$ɒ_0$
$ŧ_0$	$ŧ_0$	0	0	$ɒ_0$
$ɯ_0$	$ɯ_0$	$ɯ_0$	0	$ɒ_0$
$ɒ_0$	$ɒ_0$	$ɒ_0$	$ɒ_0$	0

Then $(Ɠ, *, 0)$ is a $\mathcal{BCK}$ -algebra. Define a NFS Ⱥ in Ɠ by
$\mathbb{P}_{Ⱥ}(0) = 0.9$, $\mathbb{P}_{Ⱥ}(ŧ_0) = \mathbb{P}_{Ⱥ}(ɯ_0) = \mathbb{P}_{Ⱥ}(ɒ_0) = 0.4$.
$ȡ_{Ⱥ}(0) = 0.9$, $ȡ_{Ⱥ}(ŧ_0) = ȡ_{Ⱥ}(ɯ_0) = ȡ_{Ⱥ}(ɒ_0) = 0.4$ and
$Ч_{Ⱥ}(0) = 0.4$, $Ч_{Ⱥ}(ŧ_0) = Ч_{Ⱥ}(ɯ_0) = Ч_{Ⱥ}(ɒ_0) = 0.9$ where 0.4 *and* $0.9 \in [0,1]$.
By usual calculations one can easily check that $Ⱥ = (\mathbb{P}_{Ⱥ}, ȡ_{Ⱥ}, Ч_{Ⱥ})$ is $\mathcal{NFPII}$ of Ɠ.

III. DNFSA and DNFI's in BCK-algebra

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Here, we employ the concept of Derivations, encompassing both (L, R)-D and (R, L)-D, to DNFSA, DNFI and initiated insight into DNFSA, DNFI and corresponding properties are analyzed.

Definition 3.1: $\Lambda: Ɠ \to Ɠ$ is a mapping that acts on Ɠ. Let $₳ = (\mathbb{P}_{₳}, ժ_{₳}, Ч_{₳})$ be a non-empty NFS of Ɠ. Then, ₳ is said to be a left derivation Neutrosophic fuzzy ideal (LDNFI) of Ɠ if it fulfills the following conditions $\forall$ ѣ, ɰ, ɣ $\in$ Ɠ:

(LDNFI-1) $\mathbb{P}_{₳}(0) \geq \mathbb{P}_{₳}(ѣ)$, $ժ_{₳}(0) \geq ժ_{₳}(ѣ)$ and $Ч_{₳}(0) \leq Ч_{₳}(ѣ)$

(LDNFI-2) $\mathbb{P}_{₳}(\Lambda(ѣ)) \geq min\{\mathbb{P}_{₳}(\Lambda(ѣ) * ɰ), \mathbb{P}_{₳}(\Lambda(ɰ))\}$

(LDNFI-3) $ժ_{₳}(\Lambda(ѣ)) \geq min\{ժ_{₳}(\Lambda(ѣ) * ɰ), ժ_{₳}(\Lambda(ɰ))\}$

(LDNFI-4) $Ч_{₳}(\Lambda(ѣ)) \leq max\{Ч_{₳}(\Lambda(ѣ) * ɰ), Ч_{₳}(\Lambda(ɰ))\}$

RDNFI of Ɠ if it fulfills :

(RDNFI-1) $\mathbb{P}_{₳}(0) \geq \mathbb{P}_{₳}(ѣ)$, $ժ_{₳}(0) \geq ժ_{₳}(ѣ)$ and $Ч_{₳}(0) \leq Ч_{₳}(ѣ)$

(RDNFI-2) $\mathbb{P}_{₳}(\Lambda(ѣ)) \geq min\{\mathbb{P}_{₳}(ѣ * \Lambda(ɰ)), \mathbb{P}_{₳}(\Lambda(ɰ))\}$

(RDNFI-3) $ժ_{₳}(\Lambda(ѣ)) \geq min\{ժ_{₳}(ѣ * \Lambda(ɰ)), ժ_{₳}(\Lambda(ɰ))\}$

(RDNFI-4) $Ч_{₳}(\Lambda(ѣ)) \leq max\{Ч_{₳}(ѣ * \Lambda(ɰ)), Ч_{₳}(\Lambda(ɰ))\}$

DNFI of Ɠ if it fulfills :

(DNFI-1) $\mathbb{P}_{₳}(0) \geq \mathbb{P}_{₳}(ѣ)$, $ժ_{₳}(0) \geq ժ_{₳}(ѣ)$ and $Ч_{₳}(0) \leq Ч_{₳}(ѣ)$

(DNFI-2) $\mathbb{P}_{₳}(\Lambda(ѣ)) \geq min\{\mathbb{P}_{₳}(\Lambda(ѣ * ɰ)), \mathbb{P}_{₳}(\Lambda(ɰ))\}$

(DNFI-3) $ժ_{₳}(\Lambda(ѣ)) \geq min\{ժ_{₳}(\Lambda(ѣ * ɰ)), ժ_{₳}(\Lambda(ɰ))\}$

(DNFI-4) $Ч_{₳}(\Lambda(ѣ)) \leq max\{Ч_{₳}(\Lambda(ѣ * ɰ)), Ч_{₳}(\Lambda(ɰ))\}$

Proposition 3. 2: Every DNFI $\mathbb{P}_{₳}, ժ_{₳}$ of Ɠ is of reversing order and $Ч_{₳}$ of Ɠ is of preserving order (or)

Let $₳ = (\mathbb{P}_{₳}, ժ_{₳}, Ч_{₳})$ be a DNFI of Ɠ. If $\Lambda(ѣ) \leq \Lambda(ɰ)$ in Ɠ, then $\mathbb{P}_{₳}(\Lambda(ѣ)) \geq \mathbb{P}_{₳}(\Lambda(ɰ))$, $ժ_{₳}(\Lambda(ѣ)) \geq ժ_{₳}(\Lambda(ɰ))$ and $Ч_{₳}(\Lambda(ѣ)) \leq Ч_{₳}(\Lambda(ɰ))$ (i.e) $\mathbb{P}_{₳}, ժ_{₳}$ of Ɠ is of reversing order and $Ч_{₳}$ of Ɠ is of preserving order.

Proof: Let $\Lambda(ѣ) \leq \Lambda(ɰ)$.

Since $\mathbb{P}_{₳}, ժ_{₳}$ are DNFI on Ɠ.

By DNFI-2, we obtain $\mathbb{P}_{₳}(\Lambda(ѣ)) \geq min\{\mathbb{P}_{₳}(\Lambda(ѣ * ɰ)), \mathbb{P}_{₳}(\Lambda(ɰ))\}$

DNFI-3, we obtain $ժ_{₳}(\Lambda(ѣ)) \geq min\{ժ_{₳}(\Lambda(ѣ * ɰ)), ժ_{₳}(\Lambda(ɰ))\}$

Since $\Lambda(ѣ) \leq \Lambda(ɰ)$, then $\Lambda(ѣ) * \Lambda(ɰ) = 0$

We know that $\Lambda(ѣ) * \Lambda(ɰ) \geq \Lambda(ѣ) * ɰ = \Lambda(ѣ * ɰ)$

Therefore,
$$\begin{aligned}\mathbb{P}_{₳}(\Lambda(ѣ)) &\geq min\{\mathbb{P}_{₳}(\Lambda(ѣ * ɰ)), \mathbb{P}_{₳}(\Lambda(ɰ))\}\\ &\geq min\{\mathbb{P}_{₳}(\Lambda(ѣ) * \Lambda(ɰ)), \mathbb{P}_{₳}(\Lambda(ɰ))\}\\ &\geq min\{\mathbb{P}_{₳}(0), \mathbb{P}_{₳}(\Lambda(ɰ))\}\\ &= \mathbb{P}_{₳}(\Lambda(ɰ))\end{aligned}$$

$$\begin{aligned}ժ_{₳}(\Lambda(ѣ)) &\geq min\{ժ_{₳}(\Lambda(ѣ * ɰ)), ժ_{₳}(\Lambda(ɰ))\}\\ &\geq min\{ժ_{₳}(\Lambda(ѣ) * \Lambda(ɰ)), ժ_{₳}(\Lambda(ɰ))\}\\ &\geq min\{ժ_{₳}(0), ժ_{₳}(\Lambda(ɰ))\}\\ &= ժ_{₳}(\Lambda(ɰ))\end{aligned}$$

By DNFI-4, we obtain
$$\begin{aligned}Ч_{₳}(\Lambda(ѣ)) &\leq max\{Ч_{₳}(\Lambda(ѣ * ɰ)), Ч_{₳}(\Lambda(ɰ))\}\\ &\leq max\{Ч_{₳}(\Lambda(ѣ) * \Lambda(ɰ)), Ч_{₳}(\Lambda(ɰ))\}\\ &\leq max\{Ч_{₳}(0), Ч_{₳}(\Lambda(ɰ))\}\\ &= Ч_{₳}(\Lambda(ɰ))\end{aligned}$$

Therefore $\mathbb{P}_{₳}(\Lambda(ѣ)) \geq \mathbb{P}_{₳}(\Lambda(ɰ))$, $ժ_{₳}(\Lambda(ѣ)) \geq ժ_{₳}(\Lambda(ɰ))$ and $Ч_{₳}(\Lambda(ѣ)) \leq Ч_{₳}(\Lambda(ɰ))$.

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IV. DNFII's of BCK-algebra

This section generalizes the derivation concept to NFIIs, introducing derivations of NFIIs and exploring their consequences, including interconnections between DNFSAs, DNFIs, and DNFII's, and related properties.

Definition 4.1: $\Lambda: Ǵ \to Ǵ$ is a mapping that acts on $Ǵ$. Consider a non-empty NFS $Ⱥ = (\mathbb{P}_{Ⱥ}, ꝺ_{Ⱥ}, Ч_{Ⱥ})$ of $Ǵ$. Then, $Ⱥ$ is said to be a left derivation Neutrosophic fuzzy implicative ideal (LDNFII) of $Ǵ$ if it fulfills the following conditions $\forall$ ѣ, щ, ɣ $\in Ǵ$:

(LDNFII-1) $\mathbb{P}_{Ⱥ}(0) \geq \mathbb{P}_{Ⱥ}(ѣ)$, $ꝺ_{Ⱥ}(0) \geq ꝺ_{Ⱥ}(ѣ)$ and $Ч_{Ⱥ}(0) \leq Ч_{Ⱥ}(ѣ)$

(LDNFII-2) $\mathbb{P}_{Ⱥ}(\Lambda(ѣ)) \geq min\,\{\mathbb{P}_{Ⱥ}((\Lambda(ѣ * (щ * ѣ))) * ɣ), \mathbb{P}_{Ⱥ}(\Lambda(ɣ))\}$

(LDNFII-3) $ꝺ_{Ⱥ}(\Lambda(ѣ)) \geq min\{ꝺ_{Ⱥ}((\Lambda(ѣ * (щ * ѣ))) * ɣ), ꝺ_{Ⱥ}(\Lambda(ɣ))\}$

(LDNFII-4) $Ч_{Ⱥ}(\Lambda(ѣ)) \leq max\,\{Ч_{Ⱥ}((\Lambda(ѣ * (щ * ѣ))) * ɣ), Ч_{Ⱥ}(\Lambda(ɣ))\}$

RDNFII of $Ǵ$ if it fulfills:

(RDNFII-1) $\mathbb{P}_{Ⱥ}(0) \geq \mathbb{P}_{Ⱥ}(ѣ)$, $ꝺ_{Ⱥ}(0) \geq ꝺ_{Ⱥ}(ѣ)$ and $Ч_{Ⱥ}(0) \leq Ч_{Ⱥ}(ѣ)$

(RDNFII-2) $\mathbb{P}_{Ⱥ}(\Lambda(ѣ)) \geq min\,\{\mathbb{P}_{Ⱥ}((ѣ * (щ * ѣ)) * \Lambda(ɣ)), \mathbb{P}_{Ⱥ}(\Lambda(ɣ))\}$

(RDNFII-3) $ꝺ_{Ⱥ}(\Lambda(ѣ)) \geq min\,\{ꝺ_{Ⱥ}((ѣ * (щ * ѣ)) * \Lambda(ɣ)), ꝺ_{Ⱥ}(\Lambda(ɣ))\}$

(RDNFII-4) $Ч_{Ⱥ}(\Lambda(ѣ)) \leq max\,\{Ч_{Ⱥ}((ѣ * (щ * ѣ)) * \Lambda(ɣ)), Ч_{Ⱥ}(\Lambda(ɣ))\}$

DNFII of $Ǵ$ if it fulfills:

(DNFII-1) $\mathbb{P}_{Ⱥ}(0) \geq \mathbb{P}_{Ⱥ}(ѣ)$, $ꝺ_{Ⱥ}(0) \geq ꝺ_{Ⱥ}(ѣ)$ and $Ч_{Ⱥ}(0) \leq Ч_{Ⱥ}(ѣ)$

(DNFII-2) $\mathbb{P}_{Ⱥ}(\Lambda(ѣ)) \geq min\,\left\{\mathbb{P}_{Ⱥ}\left(\Lambda\left((ѣ * (щ * ѣ)) * ɣ\right)\right), \mathbb{P}_{Ⱥ}(\Lambda(ɣ))\right\}$

(DNFII-3) $ꝺ_{Ⱥ}(\Lambda(ѣ)) \geq min\,\left\{ꝺ_{Ⱥ}\left(\Lambda\left((ѣ * (щ * ѣ)) * ɣ\right)\right), ꝺ_{Ⱥ}(\Lambda(ɣ))\right\}$

(DNFII-4) $Ч_{Ⱥ}(\Lambda(ѣ)) \leq max\,\left\{Ч_{Ⱥ}\left(\Lambda\left((ѣ * (щ * ѣ)) * ɣ\right)\right), Ч_{Ⱥ}(\Lambda(ɣ))\right\}$

Example 4.2: Let $Ǵ = \{0, ꝁ, ꝺ, n, ɧ\}$ be a BCK-algebra, whose binary operation is defined by the following Cayley table

*	0	ꝁ	ꝺ	n	ɧ
0	0	0	0	0	0
ꝁ	ꝁ	0	ꝁ	0	0
ꝺ	ꝺ	ꝺ	0	0	0
n	n	n	n	0	0
ɧ	ɧ	n	ɧ	ꝁ	0

Establish a mapping $\Lambda: Ǵ \to Ǵ$ by $\Lambda(ѣ) = \begin{cases} 0 \text{ if } ѣ = 0, ꝁ, ꝺ, n \\ ɧ \text{ if } ѣ = ɧ \end{cases}$

Then it follows that Λ is a derivation on $Ǵ$ and we define a NFS $Ǵ = \{0, ꝁ, ꝺ, n, ɧ\}$ $Ⱥ = (\mathbb{P}_{Ⱥ}, ꝺ_{Ⱥ}, Ч_{Ⱥ})$ in $Ǵ$ defined by $\mathbb{P}_{Ⱥ}(0) = \mathbb{P}_{Ⱥ}(ꝺ) = m_0, \mathbb{P}_{Ⱥ}(ꝁ) = \mathbb{P}_{Ⱥ}(n) = \mathbb{P}_{Ⱥ}(ɧ) = m_1$, $ꝺ_{Ⱥ}(0) = ꝺ_{Ⱥ}(ꝺ) = k_0, ꝺ_{Ⱥ}(ꝁ) = ꝺ_{Ⱥ}(n) = ꝺ_{Ⱥ}(ɧ) = k_1$ and $Ч_{Ⱥ}(0) = Ч_{Ⱥ}(ꝺ) = n_0$, $R_A(ꝁ) = Ч_{Ⱥ}(n) = Ч_{Ⱥ}(ɧ) = n_1$, where $m_i, k_j, n_k \in [0, 1]$ and $m_i + k_j + n_k \leq 1$, where $i, j, k \in [0,1]$ and suppose a derivation is defined on the NFS by $\mathbb{P}_{Ⱥ}: Ǵ \to Ǵ$, $ꝺ_{Ⱥ}: Ǵ \to Ǵ$ and $Ч_{Ⱥ}: Ǵ \to Ǵ$ such that $\mathbb{P}_{Ⱥ}(\Lambda(0)) = \mathbb{P}_{Ⱥ}(\Lambda(ꝺ)) = 1, \mathbb{P}_{Ⱥ}(\Lambda(ꝁ)) = \mathbb{P}_{Ⱥ}(\Lambda(n)) = \mathbb{P}_{Ⱥ}(\Lambda(ɧ)) = 0.5$, $ꝺ_{Ⱥ}(\Lambda(0)) = ꝺ_{Ⱥ}(\Lambda(ꝺ)) = 1, ꝺ_{Ⱥ}(\Lambda(ꝁ)) = ꝺ_{Ⱥ}(\Lambda(n)) = ꝺ_{Ⱥ}(\Lambda(ɧ)) = 0.5$ and $Ч_{Ⱥ}(\Lambda(0)) = Ч_{Ⱥ}(\Lambda(ꝺ)) = 0$, $Ч_{Ⱥ}(\Lambda(ꝁ)) = Ч_{Ⱥ}(\Lambda(n)) = Ч_{Ⱥ}(\Lambda(ɧ)) = 0.7$

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Then, a brief inspection shows that $\text{Ⱥ} = (\mathbb{P}_{\text{Ⱥ}}, \text{ƌ}_{\text{Ⱥ}}, \text{Ч}_{\text{Ⱥ}})$ is DNFII of Ɠ.

V. DNFPII of BCK-algebra

The concept of derivation is extended to NFPII's in this section, which examines their properties, derivations, and connections to DNFSA, DNFI, and DNFPII.

A NFS $\text{Ⱥ} = (\mathbb{P}_{\text{Ⱥ}}, \text{ƌ}_{\text{Ⱥ}}, \text{Ч}_{\text{Ⱥ}})$ in Ɠ is a DNFPII if it meets the following criteria:

(DNFPII-1) $\mathbb{P}_{\text{Ⱥ}}(0) \geq \mathbb{P}_{\text{Ⱥ}}(\Lambda(\text{ƀ}))$, $\text{ƌ}_{\text{Ⱥ}}(0) \geq \text{ƌ}_{\text{Ⱥ}}(\Lambda(\text{ƀ}))$ and $\text{Ч}_{\text{Ⱥ}}(0) \leq \text{Ч}_{\text{Ⱥ}}(\Lambda(\text{ƀ}))$

(DNFPII-2) $\mathbb{P}_{\text{Ⱥ}}(\boldsymbol{\Lambda}(\text{ƀ} \ast \text{ɣ})) \geq min\,\{\mathbb{P}_{\text{Ⱥ}}(\Lambda((\text{ƀ} \ast \text{ɰ}) \ast \text{ɣ})), \mathbb{P}_{\text{Ⱥ}}(\Lambda(\text{ɰ} \ast \text{ɣ}))\}$

(DNFPII-3) $\text{ƌ}_{\text{Ⱥ}}(\boldsymbol{\Lambda}(\text{ƀ} \ast \text{ɣ})) \geq min\,\{\text{ƌ}_{\text{Ⱥ}}(\Lambda((\text{ƀ} \ast \text{ɰ}) \ast \text{ɣ})), \text{ƌ}_{\text{Ⱥ}}(\Lambda(\text{ɰ} \ast \text{ɣ}))\}$

(DNFPII-4) $\text{Ч}_{\text{Ⱥ}}(\boldsymbol{\Lambda}(\text{ƀ} \ast \text{ɣ})) \leq max\,\{\text{Ч}_{\text{Ⱥ}}(\Lambda((\text{ƀ} \ast \text{ɰ}) \ast \text{ɣ})), \text{Ч}_{\text{Ⱥ}}(\Lambda(\text{ɰ} \ast \text{ɣ}))\}$,

$\forall$ ƀ, ɰ, ɣ ∈ Ɠ.

Example 5.1: Let Ɠ be a BCK-algebra with the underlying set {0, 1, 2, 3, } and the following Cayley table.

∗	0	1	2	3
0	0	0	0	0
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

Establish a mapping $\Lambda: \text{Ɠ} \to \text{Ɠ}$ by $\Lambda(\text{ƀ}) = \begin{cases} 0 \text{ if ƀ} = 0 \\ 3 \text{ if ƀ} = 1, 2, 3 \end{cases}$

Then it follows that Λ is a derivation on Ɠ and we define a NFS $\text{Ⱥ} = (\mathbb{P}_{\text{Ⱥ}}, \text{ƌ}_{\text{Ⱥ}}, \text{Ч}_{\text{Ⱥ}})$ in Ɠ by

$\mathbb{P}_{\text{Ⱥ}}(0) = 0.8, \mathbb{P}_{\text{Ⱥ}}(1) = \mathbb{P}_{\text{Ⱥ}}(2) = \mathbb{P}_{\text{Ⱥ}}(3) = 0.3$,

$\text{ƌ}_{\text{Ⱥ}}(0) = 0.8, \text{ƌ}_{\text{Ⱥ}}(1) = \text{ƌ}_{\text{Ⱥ}}(2) = \text{ƌ}_{\text{Ⱥ}}(3) = 0.3$ and

$\text{Ч}_{\text{Ⱥ}}(0) = 0.3, \text{Ч}_{\text{Ⱥ}}(1) = \text{Ч}_{\text{Ⱥ}}(2) = \text{Ч}_{\text{Ⱥ}}(3) = 0.8$ where $m_i, k_j, n_k \in [0, 1]$ and $m_i + k_j + n_k \leq 1$, where $i, j, k \in [0,1]$ and suppose a derivation is defined on the NFS by $\mathbb{P}_{\text{Ⱥ}}: \text{Ɠ} \to \text{Ɠ}$, $\text{ƌ}_{\text{Ⱥ}}: \text{Ɠ} \to \text{Ɠ}$ and $\text{Ч}_{\text{Ⱥ}}: \text{Ɠ} \to \text{Ɠ}$ such that

$\mathbb{P}_{\text{Ⱥ}}(\Lambda(0)) = 0.8, \mathbb{P}_{\text{Ⱥ}}(\Lambda(1)) = \mathbb{P}_{\text{Ⱥ}}(\Lambda(2)) = \mathbb{P}_{\text{Ⱥ}}(\Lambda(3)) = 0.3$,

$\text{ƌ}_{\text{Ⱥ}}(\Lambda(0)) = 0.8, \text{ƌ}_{\text{Ⱥ}}(\Lambda(1)) = \text{ƌ}_{\text{Ⱥ}}(\Lambda(2)) = \text{ƌ}_{\text{Ⱥ}}(\Lambda(3)) = 0.3$ and

$\text{Ч}_{\text{Ⱥ}}(\Lambda(0)) = 0.3, \text{Ч}_{\text{Ⱥ}}(\Lambda(1)) = \text{Ч}_{\text{Ⱥ}}(\Lambda(2)) = \text{Ч}_{\text{Ⱥ}}(\Lambda(3)) = 0.8$,

Evidently $\text{Ⱥ} = (\mathbb{P}_{\text{Ⱥ}}, \text{ƌ}_{\text{Ⱥ}}, \text{Ч}_{\text{Ⱥ}})$ is DNFPII of Ɠ.

Theorem 5.2: A DNFI $\text{Ⱥ} = (\mathbb{P}_{\text{Ⱥ}}, \text{ƌ}_{\text{Ⱥ}}, \text{Ч}_{\text{Ⱥ}})$ is a DNFPII if and only if it satisfies the identity for all elements ƀ, ɰ, ɣ ∈ Ɠ.

$\mathbb{P}_{\text{Ⱥ}}(\Lambda(\text{ƀ} \ast \text{ɰ})) \geq \mathbb{P}_{\text{Ⱥ}}\left(\Lambda((\text{ƀ} \ast \text{ɰ}) \ast \text{ɰ})\right)$,

$\text{ƌ}_{\text{Ⱥ}}(\Lambda(\text{ƀ} \ast \text{ɰ})) \geq \text{ƌ}_{\text{Ⱥ}}(\Lambda((\text{ƀ} \ast \text{ɰ}) \ast \text{ɰ}))$ and

$\text{Ч}_{\text{Ⱥ}}(\Lambda(\text{ƀ} \ast \text{ɰ})) \leq \text{Ч}_{\text{Ⱥ}}(\Lambda((\text{ƀ} \ast \text{ɰ}) \ast \text{ɰ}))$.

Theorem 5.3: A DNFI $\text{Ⱥ} = (\mathbb{P}_{\text{Ⱥ}}, \text{ƌ}_{\text{Ⱥ}}, \text{Ч}_{\text{Ⱥ}})$ of Ɠ is DNFPII ⇔ it satisfies the identity and

$\mathbb{P}_{\text{Ⱥ}}(\Lambda(\text{ƀ} \ast \text{ɰ})) = \mathbb{P}_{\text{Ⱥ}}(\Lambda((\text{ƀ} \ast \text{ɰ}) \ast \text{ɰ}))$,

$\text{ƌ}_{\text{Ⱥ}}(\Lambda(\text{ƀ} \ast \text{ɰ})) = \text{ƌ}_{\text{Ⱥ}}(\Lambda((\text{ƀ} \ast \text{ɰ}) \ast \text{ɰ}))$ and

$\text{Ч}_{\text{Ⱥ}}(\Lambda(\text{ƀ} \ast \text{ɰ})) = \text{Ч}_{\text{Ⱥ}}(\Lambda((\text{ƀ} \ast \text{ɰ}) \ast \text{ɰ}))$ $\forall$ ƀ, ɰ, ɣ ∈ Ɠ.

Proof: Straight Forward

Theorem 5.4: A DNFI $Ⱥ = (\mathbb{P}_{Ⱥ}, ძ_{Ⱥ}, Ч_{Ⱥ})$ of Ɠ is DNFII if and only if Ⱥ is both DNFCI and DNFPII.
Proof: Straight Forward
Theorem 5.5: Let $\mathfrak{Y} \subseteq$ Ɠ and $Ⱥ = (\mathbb{P}_{Ⱥ}, ძ_{Ⱥ}, Ч_{Ⱥ})$ be a NFS in Ɠ defined by
$\mathbb{P}_{Ⱥ}(\Lambda(ѣ)) = \begin{cases} \delta_0, ѣ \in \mathfrak{Y} \\ \delta_1, otherwise \end{cases}$, $ძ_{Ⱥ}(\Lambda(ѣ)) = \begin{cases} \eta_0, ѣ \in \mathfrak{Y} \\ \eta_1, otherwise \end{cases}$ and $Ч_{Ⱥ}(\Lambda(ѣ)) = \begin{cases} \zeta_0, ѣ \in \mathfrak{Y} \\ \zeta_1, otherwise \end{cases}$, for all $ѣ \in Ɠ$, where $0 \le \delta_1 < \delta_0$, $0 \le \eta_1 < \eta_0$ and $0 \le \zeta_0 < \zeta_1$ and $\delta_i + \eta_i + \zeta_i \le 1$, for $i = 0,1$. The next conditions are interchangeable:

(i) Ⱥ is a DNFPII of Ɠ
(ii) $\mathfrak{Y}$ is an II of Ɠ.

Proof: Let's Suppose (i)
(i.e) Ⱥ is a DNFPII of Ɠ
Let $ѣ, щ \in \mathfrak{Y}$
Now $\mathbb{P}_{Ⱥ}(0) \ge \mathbb{P}_{Ⱥ}(\Lambda(ѣ)) = \delta_0$
$\mathbb{P}_{Ⱥ}(0) \ge \delta_0$, $0 \in \mathfrak{Y}$
Let $ѣ, щ, ɣ \in Ɠ$ be such that Let $(ѣ * (щ * ɣ)) * ɣ \in \mathfrak{Y}$
We have $\mathbb{P}_{Ⱥ}(\Lambda(ѣ)) \ge min\{\mathbb{P}_{Ⱥ}(\Lambda((ѣ * (щ * ѣ)) * ɣ)), \mathbb{P}_{Ⱥ}(\Lambda(ɣ))\}$
$= min\{\delta_0, \delta_0\}$
$= \delta_0$ and so $ѣ \in \mathfrak{Y}$
Hence $\mathfrak{Y}$ is an II of Ɠ.
Let's Suppose (ii) and $ѣ \in Ɠ$
If $ѣ \in \mathfrak{Y}$ then $\mathbb{P}_{Ⱥ}(0) = \delta_0$, $\mathbb{P}_{Ⱥ}(\Lambda(ѣ)) = \delta_0$
Since $0 \in \mathfrak{Y}$ we have $\Rightarrow ძ_{Ⱥ}(0) = \eta_0$, and so $ძ_{Ⱥ}(0) = ძ_{Ⱥ}(\Lambda(ѣ))$
Also $Ч_{Ⱥ}(\Lambda(ѣ)) = \zeta_0$ and so $Ч_{Ⱥ}(0) = Ч_{Ⱥ}(\Lambda(ѣ))$
If $ѣ \notin Ɠ \Rightarrow \mathbb{P}_{Ⱥ}(\Lambda(ѣ)) = \delta_1$; $ძ_{Ⱥ}(\Lambda(ѣ)) = \eta_1$ and $Ч_{Ⱥ}(\Lambda(ѣ)) = \zeta_0$
Now $\mathbb{P}_{Ⱥ}(0) = \delta_0 > \delta_1 = \mathbb{P}_{Ⱥ}(\Lambda(ѣ))$
$ძ_{Ⱥ}(0) = \eta_0 > \eta_1 = ძ_{Ⱥ}(\Lambda(ѣ))$ and $Ч_{Ⱥ}(0) = \zeta_0 < \zeta_1 = Ч_{Ⱥ}(\Lambda(ѣ))$ $\forall ѣ \in Ɠ$.
Let $ѣ, щ, ɣ \in Ɠ$ be such that $(ѣ * (щ * ѣ)) * ɣ \in \mathfrak{Y}$, $ɣ \in Ɠ$.
If $(ѣ * (щ * ѣ)) * ɣ \in \mathfrak{Y}$ and $ɣ \in \mathfrak{Y}$,
Since $\mathfrak{Y}$ is an II of Ɠ, then we obtain $ѣ \in \mathfrak{Y}$ and so
$\mathbb{P}_{Ⱥ}(\Lambda(ѣ)) = \delta_0 = min\{\delta_0, \delta_0\} = min\{\mathbb{P}_{Ⱥ}(\Lambda((ѣ * (щ * ѣ)) * ɣ)), \mathbb{P}_{Ⱥ}(\Lambda(ɣ))\}$
$ძ_{Ⱥ}(\Lambda(ѣ)) = \eta_0 = min\{\eta_0, \eta_0\} = min\{ძ_{Ⱥ}(\Lambda((ѣ * (щ * ѣ)) * ɣ)), ძ_{Ⱥ}(\Lambda(ɣ))\}$ and
$Ч_{Ⱥ}(\Lambda(ѣ)) = \zeta_0 = max\{\zeta_0, \zeta_0\} = max\{Ч_{Ⱥ}(\Lambda((ѣ * (щ * ѣ)) * ɣ)), Ч_{Ⱥ}(\Lambda(ɣ))\}$
If $(ѣ * (щ * ѣ)) * ɣ \in \mathfrak{Y}$ and $ɣ \notin \mathfrak{Y}$, then we obtain $ѣ \notin \mathfrak{Y}$ and so
$\mathbb{P}_{Ⱥ}(\Lambda(ѣ)) = \delta_1 = min\{\delta_0, \delta_0\} = min\{\mathbb{P}_{Ⱥ}(\Lambda((ѣ * (щ * ѣ)) * ɣ)), \mathbb{P}_{Ⱥ}(\Lambda(ɣ))\}$
$ძ_{Ⱥ}(\Lambda(ѣ)) = \eta_1 = min\{\eta_0, \eta_0\} = min\{ძ_{Ⱥ}(\Lambda((ѣ * (щ * ѣ)) * ɣ)), ძ_{Ⱥ}(\Lambda(ɣ))\}$ and
$Ч_{Ⱥ}(\Lambda(ѣ)) = \zeta_1 = max\{\zeta_0, \zeta_0\} = max\{Ч_{Ⱥ}(\Lambda((ѣ * (щ * ѣ)) * ɣ)), Ч_{Ⱥ}(\Lambda(ɣ))\}$
If $(ѣ * (щ * ѣ)) * ɣ \in \mathfrak{Y}$ and $ɣ \in \mathfrak{Y}$, then we obtain $ѣ \notin \mathfrak{Y}$ and so
$\mathbb{P}_{Ⱥ}(\Lambda(ѣ)) = \delta_1 = min\{\delta_1, \delta_0\} = min\{\mathbb{P}_{Ⱥ}(\Lambda((ѣ * (щ * ѣ)) * ɣ)), \mathbb{P}_{Ⱥ}(\Lambda(ɣ))\}$
$ძ_{Ⱥ}(\Lambda(ѣ)) = \eta_1 = min\{\eta_1, \eta_0\} = min\{ძ_{Ⱥ}(\Lambda((ѣ * (щ * ѣ)) * ɣ)), ძ_{Ⱥ}(\Lambda(ɣ))\}$ and
$Ч_{Ⱥ}(\Lambda(ѣ)) = \zeta_1 = max\{\zeta_1, \zeta_0\} = max\{Ч_{Ⱥ}(\Lambda((ѣ * (щ * ѣ)) * ɣ)), Ч_{Ⱥ}(\Lambda(ɣ))\}$
If $(ѣ * (щ * ѣ)) * ɣ \in \mathfrak{Y}$ and $ɣ \notin \mathfrak{Y}$, then we obtain $ѣ \notin \mathfrak{Y}$ and so
$\mathbb{P}_{Ⱥ}(\Lambda(ѣ)) = \delta_1 = min\{\delta_1, \delta_1\} = min\{\mathbb{P}_{Ⱥ}(\Lambda((ѣ * (щ * ѣ)) * ɣ)), \mathbb{P}_{Ⱥ}(\Lambda(ɣ))\}$
$ძ_{Ⱥ}(\Lambda(ѣ)) = \eta_1 = min\{\eta_1, \eta_1\} = min\{ძ_{Ⱥ}(\Lambda((ѣ * (щ * ѣ)) * ɣ)), ძ_{Ⱥ}(\Lambda(ɣ))\}$ and

$Ч_{Ꞩ}(\Lambda(ѣ)) = \zeta_1 = max\{\zeta_1, \zeta_1\} = max\{Ч_{Ꞩ}(\Lambda((ѣ * (ɰ * ѣ)) * ɣ)), Ч_{Ꞩ}(\Lambda(ɣ))\}$

Therefore $\mathbb{P}_{Ꞩ}(\Lambda(ѣ)) \geq min\{\mathbb{P}_{Ꞩ}(\Lambda((ѣ * (ɰ * ѣ)) * ɣ)), \mathbb{P}_{Ꞩ}(\Lambda(ɣ))\}$

$ơ_{Ꞩ}(\Lambda(ѣ)) \geq min\{ơ_{Ꞩ}(\Lambda((ѣ * (ɰ * ѣ)) * ɣ)), ơ_{Ꞩ}(\Lambda(ɣ))\}$ and

$Ч_{Ꞩ}(\Lambda(ѣ)) \leq max\{Ч_{Ꞩ}(\Lambda((ѣ * (ɰ * ѣ)) * ɣ)), Ч_{Ꞩ}(\Lambda(ɣ))\}$

Therefore Ꞩ is DNFPII of Ɠ.

Corollary 5. 6: Let $\mathfrak{Y} \subseteq Ɠ$ and $Ꞩ = (\mathbb{P}_{Ꞩ}, ơ_{Ꞩ}, Ч_{Ꞩ})$ be a NFS in Ɠ defined by

$\mathbb{P}_{Ꞩ}(\Lambda(ѣ)) = \begin{cases} 1, ѣ \in \mathfrak{Y} \\ 0, otherwise \end{cases}$, $ơ_{Ꞩ}(\Lambda(ѣ)) = \begin{cases} 1, ѣ \in \mathfrak{Y} \\ 0, otherwise \end{cases}$ and

$Ч_{Ꞩ}(\Lambda(ѣ)) = \begin{cases} 0, ѣ \in \mathfrak{Y} \\ 1, otherwise \end{cases}$, for all $ѣ \in Ɠ$. Then the following statements are interchangeable:

(i) Ꞩ is a DNFII of Ɠ.

(ii) $\mathfrak{Y}$ is an II of Ɠ.

Proposition 5.7: A BCK-algebra Ɠ has the implicative property iff the same holds for all its ideals.

Theorem 5.8: Ɠ is an implicative iff every DNFI is a DNFII-type.

Proof: Ɠ is assumed to be an implicative.

As a consequence of Proposition 5.7, all ideals of Ɠ are implicative.

Suppose $Ꞩ = (\mathbb{P}_{Ꞩ}, ơ_{Ꞩ}, Ч_{Ꞩ})$ is a DNFI of Ɠ.

Then Ꞩ is DNFII of Ɠ.

On the opposite side, postulate all DNFIs of Ɠ have the DNFII property

To demonstrate that Ɠ is an implicative.

Let $\mathfrak{Y}$ be an ideal of Ɠ

Establish a NFS Ꞩ defined by

$\mathbb{P}_{Ꞩ}(ѣ) = \begin{cases} \delta_0, ѣ \in \mathfrak{Y} \\ \delta_1, otherwise \end{cases}$; $ơ_{Ꞩ}(ѣ) = \begin{cases} \eta_0, ѣ \in \mathfrak{Y} \\ \eta_1, otherwise \end{cases}$;

$Ч_{Ꞩ}(ѣ) = \begin{cases} \zeta_0, ѣ \in \mathfrak{Y} \\ \zeta_1, otherwise \end{cases}$; for all $ѣ \in Ɠ$, where $0 \leq \delta_1 < \delta_0$, $0 \leq \eta_1 < \eta_0$

and $0 \leq \zeta_0 < \zeta_1$ and

$\delta_i + \eta_i + \zeta_i \leq 1$, for $i = 0,1$.

Given that Ꞩ is a DNFI of Ɠ

we can apply Theorem 5.5 to conclude that $\mathfrak{Y}$ is an II of Ɠ.

This implies that all ideals of Ɠ are implicative.

By virtue of Proposition 5.7, we can infer that Ɠ is implicative.

Combining this outcome with the preceding propositions and theorem leads to the following corollary.

Corollary 5.9: Ɠ, as a BCK-algebra, has the following equivalent characteristics:

(i) Implicative property

(ii) All ideals are implicative

(iii) Every DNFI is a DNFII

(iv) Every DNFI is both a DNFCI and a DNFPII.

Theorem 5.10: Suppose $Ꞩ = (\mathbb{P}_{Ꞩ}, ơ_{Ꞩ}, Ч_{Ꞩ})$ is DNFI of Ɠ that fulfills the following criteria.

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(i) $\mathbb{P}_{Ⱥ}\big(Ʌ(ѣ * щ)\big) \geq min\left\{\mathbb{P}_{Ⱥ}\left(Ʌ\left(((ѣ * щ) * щ) * ɣ\right)\right), \mathbb{P}_{Ⱥ}(Ʌ(ɣ))\right\}$

(ii) $ơ_{Ⱥ}\big(Ʌ(ѣ * щ)\big) \geq min\left\{ơ_{Ⱥ}\left(Ʌ\left(((ѣ * щ) * щ) * ɣ\right)\right), ơ_{Ⱥ}(Ʌ(ɣ))\right\}$

(iii) $Ч_{Ⱥ}\big(Ʌ(ѣ * щ)\big) \leq max\left\{Ч_{Ⱥ}\left(Ʌ\left(((ѣ * щ) * щ) * ɣ\right)\right), Ч_{Ⱥ}(Ʌ(ɣ))\right\} \ \forall\, ѣ, щ, ɣ \in Ɠ.$

Then Ⱥ becomes a DNFPII ideal of Ɠ.

Proof: Suppose Ⱥ is a DNFI of Ɠ with the following constraints

$$\mathbb{P}_{Ⱥ}\big(Ʌ(ѣ * щ)\big) \geq min\left\{\mathbb{P}_{Ⱥ}\left(Ʌ\left(((ѣ * щ) * щ) * ɣ\right)\right), \mathbb{P}_{Ⱥ}(Ʌ(ɣ))\right\}$$

$$ơ_{Ⱥ}\big(Ʌ(ѣ * щ)\big) \geq min\left\{ơ_{Ⱥ}\left(Ʌ\left(((ѣ * щ) * щ) * ɣ\right)\right), ơ_{Ⱥ}(Ʌ(ɣ))\right\}$$

$$Ч_{Ⱥ}\big(Ʌ(ѣ * щ)\big) \leq max\left\{Ч_{Ⱥ}\left(Ʌ\left(((ѣ * щ) * щ) * ɣ\right)\right), Ч_{Ⱥ}(Ʌ(ɣ))\right\}$$

To prove Ⱥ is DNFPII of Ɠ.

Using $(ѣ * щ) * ɣ = (ѣ * ɣ) * щ$ and

$(ѣ * ɣ) * (щ * ɣ) \leq ѣ * ɣ$. We have

$$Ʌ\left(((ѣ * ɣ) * ɣ) * (щ * ɣ)\right) \leq Ʌ\big((ѣ * ɣ) * щ\big) = Ʌ\big((ѣ * щ) * ɣ\big) \ \forall\, ѣ, щ, ɣ \in Ɠ.$$

$$\therefore \mathbb{P}_{Ⱥ}\left(Ʌ\left(((ѣ * ɣ) * ɣ) * (щ * ɣ)\right)\right) \geq \mathbb{P}_{Ⱥ}\left(Ʌ\big((ѣ * щ) * ɣ\big)\right)$$

$$\therefore ơ_{Ⱥ}\left(Ʌ\left(((ѣ * ɣ) * ɣ) * (щ * ɣ)\right)\right) \geq ơ_{Ⱥ}\left(Ʌ\big((ѣ * щ) * ɣ\big)\right)$$

$$\therefore Ч_{Ⱥ}\left(Ʌ\left(((ѣ * ɣ) * ɣ) * (щ * ɣ)\right)\right) \leq Ч_{Ⱥ}\left(Ʌ\big((ѣ * щ) * ɣ\big)\right)$$

It follows from hypothesis,

We obtain
$$\mathbb{P}_{Ⱥ}\big(Ʌ(ѣ * ɣ)\big) \geq \min\left\{\mathbb{P}_{Ⱥ}\left(Ʌ\left(((ѣ * ɣ) * ɣ) * (щ * ɣ)\right)\right), \mathbb{P}_{Ⱥ}\big(Ʌ(щ * ɣ)\big)\right\}$$
$$\geq \min\left\{\mathbb{P}_{Ⱥ}\left(Ʌ\big((ѣ * щ) * ɣ\big)\right), \mathbb{P}_{Ⱥ}\big(Ʌ(щ * ɣ)\big)\right\}$$
$$ơ_{Ⱥ}\big(Ʌ(ѣ * ɣ)\big) \geq \min\left\{ơ_{Ⱥ}\left(Ʌ\left(((ѣ * ɣ) * ɣ) * (щ * ɣ)\right)\right), ơ_{Ⱥ}\big(Ʌ(щ * ɣ)\big)\right\}$$
$$\geq \min\left\{ơ_{Ⱥ}\left(Ʌ\big((ѣ * щ) * ɣ\big)\right), ơ_{Ⱥ}\big(Ʌ(щ * ɣ)\big)\right\}$$
And
$$Ч_{Ⱥ}\big(Ʌ(ѣ * ɣ)\big) \leq \max\left\{Ч_{Ⱥ}\left(Ʌ\left(((ѣ * ɣ) * ɣ) * (щ * ɣ)\right)\right), Ч_{Ⱥ}\big(Ʌ(щ * ɣ)\big)\right\}$$
$$\leq \max\left\{Ч_{Ⱥ}\left(Ʌ\big((ѣ * щ) * ɣ\big)\right), Ч_{Ⱥ}\big(Ʌ(щ * ɣ)\big)\right\} \ \forall\ ѣ, щ, ɣ \in Ɠ.$$

Accordingly, Ⱥ is a DNFPII of Ɠ.

Conversely, if Ⱥ is a DNFPII of Ɠ, it follows that Ⱥ is a DNFI of Ɠ.

Let $ȡ = ѣ * (щ * ɣ)$, $ɧ = ѣ * щ$

Since $Ʌ\left(\big(ѣ * (щ * ɣ)\big) * (ѣ * щ)\right) \leq Ʌ\big(щ * (щ * ɣ)\big)$

We have that

$$\mathbb{P}_{Ⱥ}\big(Ʌ((ȡ * ɧ) * ɣ)\big) = \mathbb{P}_{Ⱥ}\left(Ʌ\left(\left(\big(ѣ * (щ * ɣ)\big) * (ѣ * щ)\right) * ɣ\right)\right)$$
$$\geq \mathbb{P}_{Ⱥ}\left(Ʌ\left(\big(ѣ * (щ * ɣ)\big) * ɣ\right)\right)$$
$$= \mathbb{P}_{Ⱥ}(0)$$

and so
$$\mathbb{P}_{Ⱥ}\left(Ʌ\big((ѣ * щ) * (щ * ɣ)\big)\right) = \mathbb{P}_{Ⱥ}\big(Ʌ((ѣ * (щ * ɣ)) * ɣ)\big) = \mathbb{P}_{Ⱥ}(ȡ * ɣ)$$
$$\geq \min\left\{\mathbb{P}_{Ⱥ}\left(Ʌ\big((ȡ * ɧ) * ɣ\big)\right), \mathbb{P}_{Ⱥ}\big(Ʌ(ɧ * ɣ)\big)\right\}$$

$$= ℙ_{Ⱥ}(Λ(ђ ∗ ɣ))$$
$$= ℙ_{Ⱥ}(Λ((ѣ ∗ ɰ) ∗ ɣ))$$

$∴ \; ℙ_{Ⱥ}\left(Λ\left((ѣ ∗ ɰ) ∗ (ɰ ∗ ɣ)\right)\right) ≥ ℙ_{Ⱥ}\left(Λ\left((ѣ ∗ ɰ) ∗ ɣ\right)\right), \; ∀\, ѣ, ɰ, ɣ ∈ Ɠ.$

Similarly, We have that

$$ɗ_{Ⱥ}(Λ((ꝺ ∗ ђ) ∗ ɣ)) = ɗ_{Ⱥ}\left(Λ((ѣ ∗ (ɰ ∗ ɣ) ∗ (ѣ ∗ ɰ)) ∗ ɣ)\right)$$
$$≥ ɗ_{Ⱥ}\left(Λ\left((ѣ ∗ (ɰ ∗ ɣ)) ∗ ɣ\right)\right)$$
$$= ɗ_{Ⱥ}(0)$$

and so $ɗ_{Ⱥ}\left(Λ\left((ѣ ∗ ɰ) ∗ (ɰ ∗ ɣ)\right)\right) = ɗ_{Ⱥ}\left(Λ\left((ѣ ∗ (ɰ ∗ ɣ)\right) ∗ ɣ)\right) = ɗ_{Ⱥ}(ꝺ ∗ ɣ)$

$$≥ \min\left\{ɗ_{Ⱥ}\left(Λ\left((ꝺ ∗ ђ) ∗ ɣ\right)\right), ɗ_{Ⱥ}(Λ(ђ ∗ ɣ))\right\}$$
$$= ɗ_{Ⱥ}(Λ(ђ ∗ ɣ))$$
$$= ɗ_{Ⱥ}(Λ((ѣ ∗ ɰ) ∗ ɣ))$$
$$∴ \; ɗ_{Ⱥ}\left(Λ\left((ѣ ∗ ɰ) ∗ (ɰ ∗ ɣ)\right)\right) ≥ ɗ_{Ⱥ}(Λ((ѣ ∗ ɰ) ∗ ɣ))$$

also we have that $Ч_{Ⱥ}(Λ((ꝺ ∗ ђ) ∗ ɣ)) = Ч_{Ⱥ}\left(Λ((ѣ ∗ (ɰ ∗ ɣ) ∗ (ѣ ∗ ɰ)) ∗ ɣ)\right)$

$≤ Ч_{Ⱥ}\left(Λ\left((ѣ ∗ (ɰ ∗ ɣ)) ∗ ɣ\right)\right) = Ч_{Ⱥ}(0)$

and so $Ч_{Ⱥ}\left(Λ\left((ѣ ∗ ɰ) ∗ (ɰ ∗ ɣ)\right)\right) = Ч_{Ⱥ}\left(Λ\left((ѣ ∗ (ɰ ∗ ɣ)\right) ∗ ɣ)\right) = Ч_{Ⱥ}(ꝺ ∗ ɣ)$

$$≤ \max\left\{Ч_{Ⱥ}\left(Λ\left((ꝺ ∗ ђ) ∗ ɣ\right)\right), Ч_{Ⱥ}(Λ(ђ ∗ ɣ))\right\}$$
$$= Ч_{Ⱥ}(Λ(ђ ∗ ɣ))$$
$$= Ч_{Ⱥ}(Λ((ѣ ∗ ɰ) ∗ ɣ))$$

$∴ \; Ч_{Ⱥ}\left(Λ\left((ѣ ∗ ɰ) ∗ (ɰ ∗ ɣ)\right)\right) ≤ Ч_{Ⱥ}(Λ((ѣ ∗ ɰ) ∗ ɣ)) \; ∀\, ѣ, ɰ, ɣ ∈ Ɠ.$

Thus Proven.

Theorem 5.11: If $Ⱥ = (ℙ_{Ⱥ}, ɗ_{Ⱥ}, Ч_{Ⱥ})$ of $Ɠ$ is a DNFPII of $Ɠ$ then for any $ѣ, ɰ, ɣ, ꝺ, ђ ∈ Ɠ$.

(i) $Λ\left(\left(((ѣ ∗ ɰ) ∗ ɰ) ∗ ꝺ\right)\right) ≤ Λ(ђ)$ imply that $ℙ_{Ⱥ}(Λ(ѣ ∗ ɰ)) ≥ \min\{ℙ_{Ⱥ}(Λ(ꝺ)), ℙ_{Ⱥ}(Λ(ђ))\}$, $ɗ_{Ⱥ}(Λ(ѣ ∗ ɰ)) ≥ \min\{ɗ_{Ⱥ}(Λ(ꝺ)), ɗ_{Ⱥ}(Λ(ђ))\}$ and $Ч_{Ⱥ}(Λ(ѣ ∗ ɰ)) ≤ \max\{Ч_{Ⱥ}(Λ(ꝺ)), Ч_{Ⱥ}(Λ(ђ))\}$.

(ii) $Λ\left(\left(((ѣ ∗ ɰ) ∗ ɰ) ∗ ꝺ\right)\right) ≤ Λ(ђ)$ imply that

$ℙ_{Ⱥ}\left(Λ\left(((ѣ ∗ ɣ) ∗ (ɰ ∗ ɣ))\right)\right) ≥ \min\{ℙ_{Ⱥ}(Λ(ꝺ)), ℙ_{Ⱥ}(Λ(ђ))\}$

$ɗ_{Ⱥ}\left(Λ\left(((ѣ ∗ ɣ) ∗ (ɰ ∗ ɣ))\right)\right) ≥ \min\{ɗ_{Ⱥ}(Λ(ꝺ)), ɗ_{Ⱥ}(Λ(ђ))\}$ and

$Ч_{Ⱥ}\left(Λ\left(((ѣ ∗ ɣ) ∗ (ɰ ∗ ɣ))\right)\right) ≤ \max\{Ч_{Ⱥ}(Λ(ꝺ)), Ч_{Ⱥ}(Λ(ђ))\}$

Proof: Suppose $Ⱥ = (ℙ_{Ⱥ}, ɗ_{Ⱥ}, Ч_{Ⱥ})$ is a DNFPII of $Ɠ$.

(i) Let $ѣ, ɰ, ɣ, ꝺ, ђ ∈ Ɠ$ be such that $Λ\left(\left(((ѣ ∗ ɰ) ∗ ɰ) ∗ ꝺ\right)\right) ≤ Λ(ђ)$

We know that $ℙ_{Ⱥ}\left(Λ\left((ѣ ∗ ɰ) ∗ ɰ\right)\right) ≥ \min\{ℙ_{Ⱥ}(Λ(ꝺ)), ℙ_{Ⱥ}(Λ(ђ))\}$

$$ɗ_{Ⱥ}\left(Λ\left((ѣ ∗ ɰ) ∗ ɰ\right)\right) ≥ \min\{ɗ_{Ⱥ}(Λ(ꝺ)), ɗ_{Ⱥ}(Λ(ђ))\}$$
$$Ч_{Ⱥ}\left(Λ\left((ѣ ∗ ɰ) ∗ ɰ\right)\right) ≤ \max\{Ч_{Ⱥ}(Λ(ꝺ)), Ч_{Ⱥ}(Λ(ђ))\}$$

It follows that $ℙ_{Ⱥ}(Λ(ѣ ∗ ɰ)) ≥ \min\left\{ℙ_{Ⱥ}\left(Λ\left((ѣ ∗ ɰ) ∗ ɰ\right)\right), ℙ_{Ⱥ}\left(Λ(ɰ ∗ ɰ)\right)\right\}$

$$≥ \min\left\{ℙ_{Ⱥ}\left(Λ\left((ѣ ∗ ɰ) ∗ ɰ\right)\right), ℙ_{Ⱥ}\left(Λ(0)\right)\right\}$$

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$$\geq \min\left\{\mathbb{P}_{Ⱥ}\left(\Lambda\big((ƀ * ɰ) * ɰ\big)\right), \mathbb{P}_{Ⱥ}(0)\right\}$$
$$= \mathbb{P}_{Ⱥ}\left(\Lambda\big((ƀ * ɰ) * ɰ\big)\right)$$
$$\geq \min\{\mathbb{P}_{Ⱥ}(\Lambda(ƌ)), \mathbb{P}_{Ⱥ}(\Lambda(ɧ))\}$$
$$\therefore\ \mathbb{P}_{Ⱥ}\big(\Lambda(ƀ * ɰ)\big) \geq \min\{\mathbb{P}_{Ⱥ}(\Lambda(ƌ)), \mathbb{P}_{Ⱥ}(\Lambda(ɧ))\}$$
$$ɗ_{Ⱥ}\big(\Lambda(ƀ * ɰ)\big) \geq \min\left\{ɗ_{Ⱥ}\left(\Lambda\big((ƀ * ɰ) * ɰ\big)\right), ɗ_{Ⱥ}\big(\Lambda(ɰ * ɰ)\big)\right\}$$
$$\geq \min\left\{ɗ_{Ⱥ}\left(\Lambda\big((ƀ * ɰ) * ɰ\big)\right), ɗ_{Ⱥ}\big(\Lambda(0)\big)\right\}$$
$$\geq \min\left\{ɗ_{Ⱥ}\left(\Lambda\big((ƀ * ɰ) * ɰ\big)\right), ɗ_{Ⱥ}(0)\right\}$$
$$= ɗ_{Ⱥ}\left(\Lambda\big((ƀ * ɰ) * ɰ\big)\right)$$
$$\geq \min\{ɗ_{Ⱥ}(\Lambda(ƌ)), ɗ_{Ⱥ}(\Lambda(ɧ))\}$$
$$\therefore\ ɗ_{Ⱥ}\big(\Lambda(ƀ * ɰ)\big) \geq \min\{ɗ_{Ⱥ}(\Lambda(ƌ)), ɗ_{Ⱥ}(\Lambda(ɧ))\}$$
And
$$Ч_{Ⱥ}\big(\Lambda(ƀ * ɰ)\big) \leq \max\left\{Ч_{Ⱥ}\left(\Lambda\big((ƀ * ɰ) * ɰ\big)\right), Ч_{Ⱥ}\big(\Lambda(ɰ * ɰ)\big)\right\}$$
$$\leq \max\left\{Ч_{Ⱥ}\left(\Lambda\big((ƀ * ɰ) * ɰ\big)\right), Ч_{Ⱥ}\big(\Lambda(0)\big)\right\}$$
$$\leq \max\left\{Ч_{Ⱥ}\left(\Lambda\big((ƀ * ɰ) * ɰ\big)\right), Ч_{Ⱥ}(0)\right\}$$
$$= Ч_{Ⱥ}\left(\Lambda\big((ƀ * ɰ) * ɰ\big)\right)$$
$$\leq \max\{Ч_{Ⱥ}(\Lambda(ƌ)), Ч_{Ⱥ}(\Lambda(ɧ))\}$$
$$\therefore\ Ч_{Ⱥ}\big(\Lambda(ƀ * ɰ)\big) \leq \max\{Ч_{Ⱥ}(\Lambda(ƌ)), Ч_{Ⱥ}(\Lambda(ɧ))\}$$

(ii) Let $ƀ, ɰ, ɣ \in Ɠ$ be such that $\Lambda\left(\big(((ƀ * ɰ) * ɰ\big) * ƌ\big)\right) \leq \Lambda(ɧ)$

Since $Ⱥ$ is DNFPII of $Ɠ$.

we get $\mathbb{P}_{Ⱥ}\left(\Lambda\big(((ƀ * ɣ) * (ɰ * ɣ))\big)\right) \geq \min\{\mathbb{P}_{Ⱥ}(\Lambda(ƌ)), \mathbb{P}_{Ⱥ}(\Lambda(ɧ))\}$

$ɗ_{Ⱥ}\left(\Lambda\big(((ƀ * ɣ) * (ɰ * ɣ))\big)\right) \geq \min\{ɗ_{Ⱥ}(\Lambda(ƌ)), ɗ_{Ⱥ}(\Lambda(ɧ))\}$ and

$Ч_{Ⱥ}\left(\Lambda\big(((ƀ * ɣ) * (ɰ * ɣ))\big)\right) \leq \max\{Ч_{Ⱥ}(\Lambda(ƌ)), Ч_{Ⱥ}(\Lambda(ɧ))\}$

Therefore, the proof is complete.

Theorem 5.12: Consider $Ⱥ = (\mathbb{P}_{Ⱥ}, ɗ_{Ⱥ}, Ч_{Ⱥ})$ a NFS in $Ɠ$ that fulfills the following conditions, $\Lambda\left(\big((ƀ * ɰ) * ɰ\big) * ƌ\right) \leq \Lambda(ɧ)$ imply that $\mathbb{P}_{Ⱥ}(\Lambda(ƀ * ɰ)) \geq \min\{\mathbb{P}_{Ⱥ}(\Lambda(ƌ)), \mathbb{P}_{Ⱥ}(\Lambda(ɧ))\}$, $ɗ_{Ⱥ}\big(\Lambda(ƀ * ɰ)\big) \geq \min\{ɗ_{Ⱥ}\big(\Lambda(ƌ)\big), ɗ_{Ⱥ}\big(\Lambda(ɧ)\big)\}$ and $Ч_{Ⱥ}\big(\Lambda(ƀ * ɰ)\big) \leq \max\{Ч_{Ⱥ}(\Lambda(ƌ)), Ч_{Ⱥ}(\Lambda(ɧ))\}$, for any $ƀ, ɰ, ɣ, ƌ, ɧ \in Ɠ$. Then $Ⱥ$ has the DNFPII property in $Ɠ$.

VI. Neutrosophic Fuzzy Translations of Positive Implicative Ideals of $\mathcal{BCK}$ -algebras

In this phase, we introduce and practice the idea of Fuzzy Translations (FT) to Neutrosophic fuzzy Positive Implicative ideals in BCK-algebras and few properties are examined.

Definition 6.1: Let $Ⱥ = (\mathbb{P}_{Ⱥ}, ɗ_{Ⱥ}, Ч_{Ⱥ})$ be a NFS of $Ɠ$ and let $\beta \in [0, C]$. An object having the form $Ⱥ_{\beta}^{T} = \big((\mathbb{P}_{Ⱥ})_{\beta}^{T}, (ɗ_{Ⱥ})_{\beta}^{T}, (Ч_{Ⱥ})_{\beta}^{T}\big)$ is called a $NF^{\beta} - T$(Neutrosophic fuzzy β-translation) of $Ⱥ$ if $(\mathbb{P}_{Ⱥ})_{\beta}^{T}(ƀ) = \mathbb{P}_{Ⱥ}(ƀ) + \beta$, $(ɗ_{Ⱥ})_{\beta}^{T}(ƀ) = ɗ_{Ⱥ}(ƀ) + \beta$ and $(Ч_{Ⱥ})_{\beta}^{T}(ƀ) = Ч_{Ⱥ}(ƀ) - \beta\ \forall\, ƀ \in Ɠ$.

For notational convenience, $Ⱥ$ is represented as $Ⱥ = (\mathbb{P}_{Ⱥ}, ɗ_{Ⱥ}, Ч_{Ⱥ})$.

Theorem 6.2. For any $\mathcal{NFPII}$ $Ⱥ = (ℙ_{Ⱥ}, ɗ_{Ⱥ}, Ч_{Ⱥ})$ of $Ǵ$, the $NF^{\beta} - TȺ_{\beta}^{T}$ of $Ⱥ$ of $Ǵ$ is also $\mathcal{NFPII}$ for all $\beta \in [0, C]$.

Proof: Given that $Ⱥ = (ℙ_{Ⱥ}, ɗ_{Ⱥ}, Ч_{Ⱥ})$ is a $\mathcal{NFPII}$ of $Ǵ$, We have

$(ℙ_{Ⱥ})_{\beta}^{T}(0) = ℙ_{Ⱥ}(0) + \beta \geq ℙ_{Ⱥ}(ѣ) + \beta = (ℙ_{Ⱥ})_{\beta}^{T}(ѣ)$

$(ɗ_{Ⱥ})_{\beta}^{T}(0) = ɗ_{Ⱥ}(0) + \beta \geq ɗ_{Ⱥ}(ѣ) + \beta = (ɗ_{Ⱥ})_{\beta}^{T}(ѣ)$ and

$(Ч_{Ⱥ})_{\beta}^{T}(0) = Ч_{Ⱥ}(0) - \beta \leq Ч_{Ⱥ}(ѣ) - \beta = (Ч_{Ⱥ})_{\beta}^{T}(ѣ)$.

Now, $(ℙ_{Ⱥ})_{\beta}^{T}(ѣ * ɣ) = ℙ_{Ⱥ}(ѣ * ɣ) + \beta \geq min\{ℙ_{Ⱥ}((ѣ * ɰ) * ɣ), ℙ_{Ⱥ}(ɰ * ɣ)\} + \beta$

$= min\{(ℙ_{Ⱥ})_{\beta}^{T}((ѣ * ɰ) * ɣ), (ℙ_{Ⱥ})_{\beta}^{T}(ɰ * ɣ)\}$

$(ɗ_{Ⱥ})_{\beta}^{T}(ѣ * ɣ) = ɗ_{Ⱥ}(ѣ * ɣ) + \beta \geq min\{ɗ_{Ⱥ}((ѣ * ɰ) * ɣ), ɗ_{Ⱥ}(ɰ * ɣ)\} + \beta$

$= min\{(ɗ_{Ⱥ})_{\beta}^{T}((ѣ * ɰ) * ɣ), (ɗ_{Ⱥ})_{\beta}^{T}(ɰ * ɣ)\}$ and

$(Ч_{Ⱥ})_{\beta}^{T}(ѣ * ɣ) = Ч_{Ⱥ}(ѣ * ɣ) - \beta \leq max\{Ч_{Ⱥ}((ѣ * ɰ) * ɣ), Ч_{Ⱥ}(ɰ * ɣ)\} - \beta$

$= max\{(Ч_{Ⱥ})_{\beta}^{T}((ѣ * ɰ) * ɣ), (Ч_{Ⱥ})_{\beta}^{T}(ɰ * ɣ)\}$

It follows that, $(ℙ_{Ⱥ})_{\beta}^{T}(ѣ * ɣ) \geq min\{(ℙ_{Ⱥ})_{\beta}^{T}((ѣ * ɰ) * ɣ), (ℙ_{Ⱥ})_{\beta}^{T}(ɰ * ɣ)\}$

$(ɗ_{Ⱥ})_{\beta}^{T}(ѣ * ɣ) \geq min\{(ɗ_{Ⱥ})_{\beta}^{T}((ѣ * ɰ) * ɣ), (ɗ_{Ⱥ})_{\beta}^{T}(ɰ * ɣ)\}$ and

$(Ч_{Ⱥ})_{\beta}^{T}(ѣ * ɣ) \leq max\{(Ч_{Ⱥ})_{\beta}^{T}((ѣ * ɰ) * ɣ), (Ч_{Ⱥ})_{\beta}^{T}(ɰ * ɣ)\}$ $\forall\, ѣ, ɰ, ɣ \in Ǵ$.

Therefore, the $NF^{\beta} - TȺ_{\beta}^{T}$ of $Ⱥ$ is a $\mathcal{NFPII}$ of $Ǵ$.

Theorem 6.3. Let $Ⱥ = (ℙ_{Ⱥ}, ɗ_{Ⱥ}, Ч_{Ⱥ})$ be a NFS of $Ǵ$ such that the $NF^{\beta} - TȺ_{\beta}^{T}$ of $Ⱥ$ is a $\mathcal{NFPII}$ of $Ǵ$ for some $\beta \in [0, C]$. Then $Ⱥ$ is $\mathcal{NFPII}$ of $Ǵ$.

Proof: Consider the case where $Ⱥ_{\beta}^{T}$ is a $\mathcal{NFPII}$ of $Ǵ$ for some $\beta \in [0, C]$.

Let $ѣ, ɰ, ɣ \in Ǵ$. We have $(ℙ_{Ⱥ})_{\beta}^{T}(0) = ℙ_{Ⱥ}(0) + \beta \geq ℙ_{Ⱥ}(ѣ) + \beta = (ℙ_{Ⱥ})_{\beta}^{T}(ѣ)$

$(ɗ_{Ⱥ})_{\beta}^{T}(0) = ɗ_{Ⱥ}(0) + \beta \geq ɗ_{Ⱥ}(ѣ) + \beta = (ɗ_{Ⱥ})_{\beta}^{T}(ѣ)$ and

$(Ч_{Ⱥ})_{\beta}^{T}(0) = Ч_{Ⱥ}(0) - \beta \leq Ч_{Ⱥ}(ѣ) - \beta = (Ч_{Ⱥ})_{\beta}^{T}(ѣ)$

This leads to $ℙ_{Ⱥ}(0) \geq ℙ_{Ⱥ}(ѣ)$, $ɗ_{Ⱥ}(0) \geq ɗ_{Ⱥ}(ѣ)$ and $Ч_{Ⱥ}(0) \leq Ч_{Ⱥ}(ѣ)$.

Presently, we observe

$ℙ_{Ⱥ}(ѣ * ɣ) + \beta = (ℙ_{Ⱥ})_{\beta}^{T}(ѣ * ɣ) \geq min\{(ℙ_{Ⱥ})_{\beta}^{T}((ѣ * ɰ) * ɣ), (ℙ_{Ⱥ})_{\beta}^{T}(ɰ * ɣ)\}$

$= min\{ℙ_{Ⱥ}((ѣ * ɰ) * ɣ) + \beta, ℙ_{Ⱥ}(ɰ * ɣ) + \beta\}$

$= min\{ℙ_{Ⱥ}((ѣ * ɰ) * ɣ), ℙ_{Ⱥ}(ɰ * ɣ)\} + \beta$

$ɗ_{Ⱥ}(ѣ * ɣ) + \beta = (ɗ_{Ⱥ})_{\beta}^{T}(ѣ * ɣ) \geq min\{(ɗ_{Ⱥ})_{\beta}^{T}((ѣ * ɰ) * ɣ), (ɗ_{Ⱥ})_{\beta}^{T}(ɰ * ɣ)\}$

$= min\{ɗ_{Ⱥ}((ѣ * ɰ) * ɣ) + \beta, ɗ_{Ⱥ}(ɰ * ɣ) + \beta\}$

$= min\{ɗ_{Ⱥ}((ѣ * ɰ) * ɣ), ɗ_{Ⱥ}(ɰ * ɣ)\} + \beta$ and

$Ч_{Ⱥ}(ѣ * ɣ) - \beta = (Ч_{Ⱥ})_{\beta}^{T}(ѣ * ɣ) \leq max\{(Ч_{Ⱥ})_{\beta}^{T}((ѣ * ɰ) * ɣ), (Ч_{Ⱥ})_{\beta}^{T}(ɰ * ɣ)\}$

$= max\{Ч_{Ⱥ}((ѣ * ɰ) * ɣ) + \beta, Ч_{Ⱥ}(ɰ * ɣ) - \beta\}$

$= max\{Ч_{Ⱥ}((ѣ * ɰ) * ɣ), Ч_{Ⱥ}(ɰ * ɣ)\} - \beta$

This yields $ℙ_{Ⱥ}(ѣ * ɣ) \geq min\{ℙ_{Ⱥ}((ѣ * ɰ) * ɣ), ℙ_{Ⱥ}(ɰ * ɣ)\}$

$ɗ_{Ⱥ}(ѣ * ɣ) \geq min\{ɗ_{Ⱥ}((ѣ * ɰ) * ɣ), ɗ_{Ⱥ}(ɰ * ɣ)\}$ and

$Ч_{Ⱥ}(ѣ * ɣ) \leq max\{Ч_{Ⱥ}((ѣ * ɰ) * ɣ), Ч_{Ⱥ}(ɰ * ɣ)\}$ $\forall\, ѣ, ɰ, ɣ \in Ǵ$.

Therefore, we can deduce that $Ⱥ$ is $\mathcal{NFPII}$ of $Ǵ$.

Theorem 6.4 If the $NF^{\beta} - TȺ_{\beta}^{T}$ induced by $Ⱥ$ is a $\mathcal{NFPII}$ of $Ǵ$ for all $\beta \in [0, C]$, then it must be $\mathcal{NFI}$ of $Ǵ$.

Proof: Let the $NF^{\beta} - TȺ_{\beta}^{T}$ of $Ⱥ$ is a $\mathcal{NFPII}$ of $Ǵ$, Sequently, we obtain

$(ℙ_{Ⱥ})^T_\beta(ѣ * ɣ) \geq min\{(ℙ_{Ⱥ})^T_\beta((ѣ * ɰ) * ɣ), (ℙ_{Ⱥ})^T_\beta(ɰ * ɣ)\}$
$(ɗ_{Ⱥ})^T_\beta(ѣ * ɣ) \geq min\{(ɗ_{Ⱥ})^T_\beta((ѣ * ɰ) * ɣ), (ɗ_{Ⱥ})^T_\beta(ɰ * ɣ)\}$ and
$(Ч_{Ⱥ})^T_\beta(ѣ * ɣ) \leq max\{(Ч_{Ⱥ})^T_\beta((ѣ * ɰ) * ɣ), (Ч_{Ⱥ})^T_\beta(ɰ * ɣ)\}\ \forall\ ѣ, ɰ, ɣ \in Ɠ$.
Given any $ѣ \in Ɠ$, $ѣ * 0 = ѣ$, thus with the setting of $ɣ = 0$ we attain
$(ℙ_{Ⱥ})^T_\beta(ѣ * 0) \geq min\{(ℙ_{Ⱥ})^T_\beta((ѣ * ɰ) * 0), (ℙ_{Ⱥ})^T_\beta(ɰ * 0)\}$
$\Rightarrow (ℙ_{Ⱥ})^T_\beta(ѣ) \geq min\{(ℙ_{Ⱥ})^T_\beta(ѣ * ɰ), (ℙ_{Ⱥ})^T_\beta(ɰ)\}$
$(ɗ_{Ⱥ})^T_\beta(ѣ * 0) \geq min\{(ɗ_{Ⱥ})^T_\beta((ѣ * ɰ) * 0), (ɗ_{Ⱥ})^T_\beta(ɰ * 0)\}$
$\Rightarrow (ɗ_{Ⱥ})^T_\beta(ѣ) \geq min\{(ɗ_{Ⱥ})^T_\beta(ѣ * ɰ), (ɗ_{Ⱥ})^T_\beta(ɰ)\}$ and
$(Ч_{Ⱥ})^T_\beta(ѣ * 0) \leq max\{(Ч_{Ⱥ})^T_\beta((ѣ * ɰ) * 0), (Ч_{Ⱥ})^T_\beta(ɰ * 0)\}$
$\Rightarrow (Ч_{Ⱥ})^T_\beta(ѣ) \leq max\{(Ч_{Ⱥ})^T_\beta(ѣ * ɰ), (Ч_{Ⱥ})^T_\beta(ɰ)\}\ \forall\ ѣ, ɰ, ɣ \in Ɠ$.
Accordingly, $Ⱥ^T_\beta$ is a $\mathcal{NFI}$ of $Ɠ$.

Remark 6.5 The converse of Theorem 6.4 does not necessarily hold, as illustrated by the following counterexample.

Example 6.6 Let $Ɠ = \{0, ŧ, 𝔴, 𝔡, 𝔠\}$ be a BCK-algebra with the given table

*	0	ŧ	𝔴	𝔡	𝔠
0	0	0	0	0	0
ŧ	ŧ	0	ŧ	0	0
𝔴	𝔴	𝔴	0	0	0
𝔡	𝔡	𝔡	𝔡	0	0
𝔠	𝔠	𝔠	𝔠	𝔡	0

Define a NFS $Ⱥ$ in $Ɠ$ by
$(ℙ_{Ⱥ})^T_\beta(0) = 0.64$, $(ℙ_{Ⱥ})^T_\beta(ŧ) = 0.55$, $(ℙ_{Ⱥ})^T_\beta(𝔴) = (ℙ_{Ⱥ})^T_\beta(𝔡) = (ℙ_{Ⱥ})^T_\beta(𝔠) = 0.35$
$(ɗ_{Ⱥ})^T_\beta(0) = 0.64$, $(ɗ_{Ⱥ})^T_\beta(ŧ) = 0.55$, $(ɗ_{Ⱥ})^T_\beta(𝔴) = (ɗ_{Ⱥ})^T_\beta(𝔡) = (ɗ_{Ⱥ})^T_\beta(𝔠) = 0.35$ and
$(Ч_{Ⱥ})^T_\beta(0) = 0.55$, $(Ч_{Ⱥ})^T_\beta(ŧ) = 0.62$, $(Ч_{Ⱥ})^T_\beta(𝔴) = (Ч_{Ⱥ})^T_\beta(𝔡) = (Ч_{Ⱥ})^T_\beta(𝔠) = 0.82$.
Here, $C = 0.35$. let us take $\beta = 0.32$ then $Ⱥ^T_\beta$ of $Ⱥ$ is given by
$(ℙ_{Ⱥ})^T_\beta(0) = 0.96$, $(ℙ_{Ⱥ})^T_\beta(ŧ) = 0.87$, $(ℙ_{Ⱥ})^T_\beta(𝔴) = (ℙ_{Ⱥ})^T_\beta(𝔡) = (ℙ_{Ⱥ})^T_\beta(𝔠) = 0.67$
$(ɗ_{Ⱥ})^T_\beta(0) = 0.96$, $(ɗ_{Ⱥ})^T_\beta(ŧ) = 0.87$, $(ɗ_{Ⱥ})^T_\beta(𝔴) = (ɗ_{Ⱥ})^T_\beta(𝔡) = (ɗ_{Ⱥ})^T_\beta(𝔠) = 0.67$ and
$(Ч_{Ⱥ})^T_\beta(0) = 0.23$, $(Ч_{Ⱥ})^T_\beta(ŧ) = 0.30$, $(Ч_{Ⱥ})^T_\beta(𝔴) = (Ч_{Ⱥ})^T_\beta(𝔡) = (Ч_{Ⱥ})^T_\beta(𝔠) = 0.50$
By direct computation, we find that $Ⱥ^T_\beta$ is indeed a $\mathcal{NFI}$ of $Ɠ$.
However, it is not a $\mathcal{NFPII}$ of $Ɠ$, because
$(ℙ_{Ⱥ})^T_\beta(𝔠 * 𝔡) = 0.67 < 0.96 = min\{(ℙ_{Ⱥ})^T_\beta((𝔠 * 𝔡) * 𝔡), (ℙ_{Ⱥ})^T_\beta(𝔡 * 𝔡)\}$
$(ɗ_{Ⱥ})^T_\beta(𝔠 * 𝔡) = 0.67 < 0.96 = min\{(ɗ_{Ⱥ})^T_\beta((𝔠 * 𝔡) * 𝔡), (ɗ_{Ⱥ})^T_\beta(𝔡 * 𝔡)\}$ and
$(Ч_{Ⱥ})^T_\beta(𝔠 * 𝔡) = 0.50 > 0.23 = max\{(Ч_{Ⱥ})^T_\beta((𝔠 * 𝔡) * 𝔡), (Ч_{Ⱥ})^T_\beta(𝔡 * 𝔡)\}$.

Theorem 6.7 Let $Ⱥ = (ℙ_{Ⱥ}, ɗ_{Ⱥ}, Ч_{Ⱥ})$ be a NFS such that $NF^\beta - TȺ^T_\beta$ of $Ⱥ$ is a $\mathcal{NFPII}$ of $Ɠ\ \forall\ \beta \in [0, C]$ then the sets $\mathfrak{F} = \{𝔡 | 𝔡 \in Ɠ, (ℙ_{Ⱥ})^T_\beta(𝔡) = (ℙ_{Ⱥ})^T_\beta(0)\}$,$\mathfrak{G} = \{𝔡 | 𝔡 \in Ɠ, (ɗ_{Ⱥ})^T_\beta(𝔡) = (ɗ_{Ⱥ})^T_\beta(0)\}$ and $\mathfrak{H} = \{𝔡 | 𝔡 \in Ɠ, (Ч_{Ⱥ})^T_\beta(𝔡) = (Ч_{Ⱥ})^T_\beta(0)\}$ are PII's of $Ɠ$.
Proof: Assume that $Ⱥ^T_\beta$ is a $\mathcal{NFPII}$ of $Ɠ$. Then $(ℙ_{Ⱥ})^T_\beta$, $(ɗ_{Ⱥ})^T_\beta$ and $(Ч_{Ⱥ})^T_\beta$ are PII's of $Ɠ$.
It is evident that $0 \in \mathfrak{F}$, $0 \in \mathfrak{G}$ and $0 \in \mathfrak{H}$.

Thus $\mathfrak{F} \neq \emptyset$, $\mathfrak{G} \neq \emptyset$ and $\mathfrak{H} \neq \emptyset$.
For $(ѣ * щ) * ɣ \in \mathfrak{F}$ and $щ * ɣ \in \mathfrak{F}$ implies
$(\mathbb{P}_{A})^{T}_{\beta}((ѣ * щ) * ɣ) = (\mathbb{P}_{A})^{T}_{\beta}(0) = (\mathbb{P}_{A})^{T}_{\beta}(щ * ɣ)$
We now turn to $(\mathbb{P}_{A})^{T}_{\beta}(ѣ * ɣ) \geq min\{(\mathbb{P}_{A})^{T}_{\beta}((ѣ * щ) * ɣ), (\mathbb{P}_{A})^{T}_{\beta}(щ * ɣ)\}$
$= min\{(\mathbb{P}_{A})^{T}_{\beta}(0), (\mathbb{P}_{A})^{T}_{\beta}(0)\} = (\mathbb{P}_{A})^{T}_{\beta}(0)$
Which entails $(\mathbb{P}_{A})^{T}_{\beta}(ѣ * ɣ) \geq (\mathbb{P}_{A})^{T}_{\beta}(0)$
This shows that $\mathbb{P}_{A}(ѣ * ɣ) + \beta \geq \mathbb{P}_{A}(0) + \beta$ or $\mathbb{P}_{A}(ѣ * ɣ) \geq \mathbb{P}_{A}(0)$
In order that $ѣ * ɣ \in \mathfrak{F}$, $\forall ѣ, щ, ɣ \in Ǵ$.
Thus $\mathfrak{F}$ is PII of Ǵ. Using a similar approach we can prove $\mathfrak{G}$ and $\mathfrak{H}$ are PII's of Ǵ.

Lemma 6.8 Let the $NF^{\beta} - TA^{T}_{\beta}$ of A be a $\mathcal{NFPII}$ of Ǵ for all $\beta \in [0, C]$ then
$ѣ \leq щ \Rightarrow (\mathbb{P}_{A})^{T}_{\beta}(ѣ) \geq (\mathbb{P}_{A})^{T}_{\beta}(щ)$, $(ძ_{A})^{T}_{\beta}(ѣ) \geq (ძ_{A})^{T}_{\beta}(щ)$ and $(Ⴗ_{A})^{T}_{\beta}(ѣ) \leq (Ⴗ_{A})^{T}_{\beta}(щ)$
$\forall ѣ, щ \in Ǵ$.
Proof: Let $ѣ, щ \in Ǵ$ such that $ѣ \leq щ \Rightarrow ѣ * щ = 0$.
Consider $(\mathbb{P}_{A})^{T}_{\beta}(ѣ) = (\mathbb{P}_{A})^{T}_{\beta}(ѣ * 0) \geq min\{(\mathbb{P}_{A})^{T}_{\beta}((ѣ * щ) * 0), (\mathbb{P}_{A})^{T}_{\beta}(щ * 0)\}$
$= min\{(\mathbb{P}_{A})^{T}_{\beta}(ѣ * щ), (\mathbb{P}_{A})^{T}_{\beta}(щ)\} = min\{(\mathbb{P}_{A})^{T}_{\beta}(0), (\mathbb{P}_{A})^{T}_{\beta}(щ)\} = (\mathbb{P}_{A})^{T}_{\beta}(щ)$.
Likewise $(ძ_{A})^{T}_{\beta}(ѣ) = (ძ_{A})^{T}_{\beta}(ѣ * 0) \geq min\{(ძ_{A})^{T}_{\beta}((ѣ * щ) * 0), (ძ_{A})^{T}_{\beta}(щ * 0)\}$
$= min\{(ძ_{A})^{T}_{\beta}(ѣ * щ), (ძ_{A})^{T}_{\beta}(щ)\} = min\{(ძ_{A})^{T}_{\beta}(0), (ძ_{A})^{T}_{\beta}(щ)\} = (ძ_{A})^{T}_{\beta}(щ)$ and
$(Ⴗ_{A})^{T}_{\beta}(ѣ) = (Ⴗ_{A})^{T}_{\beta}(ѣ * 0) \leq max\{(Ⴗ_{A})^{T}_{\beta}((ѣ * щ) * 0), (Ⴗ_{A})^{T}_{\beta}(щ * 0)\}$
$= max\{(Ⴗ_{A})^{T}_{\beta}(ѣ * щ), (Ⴗ_{A})^{T}_{\beta}(щ)\} = max\{(Ⴗ_{A})^{T}_{\beta}(0), (Ⴗ_{A})^{T}_{\beta}(щ)\} = (Ⴗ_{A})^{T}_{\beta}(щ)$.
Therefore, $(\mathbb{P}_{A})^{T}_{\beta}(ѣ) \geq (\mathbb{P}_{A})^{T}_{\beta}(щ)$, $(ძ_{A})^{T}_{\beta}(ѣ) \geq (ძ_{A})^{T}_{\beta}(щ)$ and $(Ⴗ_{A})^{T}_{\beta}(ѣ) \leq (Ⴗ_{A})^{T}_{\beta}(щ)$
$\forall ѣ, щ \in Ǵ$. Hence, the result.

Theorem 6.9 If Ǵ is PI BCK-algebra, then a $\mathcal{NFI}$ must be a $\mathcal{NFPII}$.
Proof: Suppose A^{T}_{β} is a $\mathcal{NFI}$ of Ǵ and Ǵ is a PI, by definition
$(ѣ * щ) * ɣ = (ѣ * ɣ) * (щ * ɣ)$, $\forall ѣ, щ, ɣ \in Ǵ$.
Since A^{T}_{β} is a $\mathcal{NFI}$ of Ǵ. Put $ѣ * ɣ$ in place of ѣ and $щ * ɣ$ in place of щ in $\mathcal{NFI}$ -2,3 and 4.
We obtain $(\mathbb{P}_{A})^{T}_{\beta}(ѣ * ɣ) \geq min\{(\mathbb{P}_{A})^{T}_{\beta}((ѣ * ɣ) * (щ * ɣ)), (\mathbb{P}_{A})^{T}_{\beta}(щ * ɣ)\}$
$= min\{(\mathbb{P}_{A})^{T}_{\beta}((ѣ * щ) * ɣ), (\mathbb{P}_{A})^{T}_{\beta}(щ * ɣ)\}$
$(ძ_{A})^{T}_{\beta}(ѣ * ɣ) \geq min\{(ძ_{A})^{T}_{\beta}((ѣ * ɣ) * (щ * ɣ)), (ძ_{A})^{T}_{\beta}(щ * ɣ)\}$
$= min\{(ძ_{A})^{T}_{\beta}((ѣ * щ) * ɣ), (ძ_{A})^{T}_{\beta}(щ * ɣ)\}$ and
$(Ⴗ_{A})^{T}_{\beta}(ѣ * ɣ) \leq max\{(Ⴗ_{A})^{T}_{\beta}((ѣ * ɣ) * (щ * ɣ)), (Ⴗ_{A})^{T}_{\beta}(щ * ɣ)\}$
$= max\{(Ⴗ_{A})^{T}_{\beta}((ѣ * щ) * ɣ), (Ⴗ_{A})^{T}_{\beta}(щ * ɣ)\}$.
Therefore, A^{T}_{β} is a $\mathcal{NFPII}$ of Ǵ.

Theorem 6.10 Let A^{T}_{β} be a $\mathcal{NFI}$ of Ǵ then A^{T}_{β} is a $\mathcal{NFPII}$ of Ǵ, the following inequalities hold:
$(\mathbb{P}_{A})^{T}_{\beta}((ѣ * ɣ) * (щ * ɣ)) \geq (\mathbb{P}_{A})^{T}_{\beta}((ѣ * щ) * ɣ)$,
$(ძ_{A})^{T}_{\beta}((ѣ * ɣ) * (щ * ɣ)) \geq (ძ_{A})^{T}_{\beta}((ѣ * щ) * ɣ)$ and
$(Ⴗ_{A})^{T}_{\beta}((ѣ * ɣ) * (щ * ɣ)) \leq (Ⴗ_{A})^{T}_{\beta}((ѣ * щ) * ɣ)$ $\forall ѣ, щ, ɣ \in Ǵ$.
Proof: Assume that A^{T}_{β} is a $\mathcal{NFPII}$ of Ǵ.
By Theorem 6.4, A^{T}_{β} be a $\mathcal{NFI}$ of Ǵ. Let $ѣ, щ, ɣ \in Ǵ$ and $\mathscr{p} = ѣ * (щ * ɣ)$ and $\mathscr{r} = ѣ * щ$.
Since, for all $ѣ, щ, ɣ \in Ǵ$,

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$(\mathbb{P}_{Ⱥ})^{T}_{\beta}\left(\left((ѣ * (ɰ * ɣ)) * (ѣ * ɰ)\right) * ɣ\right) \geq (\mathbb{P}_{Ⱥ})^{T}_{\beta}\left((ɰ * (ɰ * ɣ)) * ɣ\right)$

$(ɗ_{Ⱥ})^{T}_{\beta}\left(\left((ѣ * (ɰ * ɣ)) * (ѣ * ɰ)\right) * ɣ\right) \geq (ɗ_{Ⱥ})^{T}_{\beta}\left((ɰ * (ɰ * ɣ)) * ɣ\right)$ and

$(Ч_{Ⱥ})^{T}_{\beta}\left(\left((ѣ * (ɰ * ɣ)) * (ѣ * ɰ)\right) * ɣ\right) \leq (Ч_{Ⱥ})^{T}_{\beta}\left((ɰ * (ɰ * ɣ)) * ɣ\right)$

Then $(\mathbb{P}_{Ⱥ})^{T}_{\beta}((\wp * r) * ɣ) \geq (\mathbb{P}_{Ⱥ})^{T}_{\beta}\left((ɰ * (ɰ * ɣ)) * ɣ\right)$

$= (\mathbb{P}_{Ⱥ})^{T}_{\beta}((ɰ * ɣ) * (ɰ * ɣ))$ [by P-3]

$= (\mathbb{P}_{Ⱥ})^{T}_{\beta}(0)$ [by BCK-3]

so $(\mathbb{P}_{Ⱥ})^{T}_{\beta}((\wp * r) * ɣ) = (\mathbb{P}_{Ⱥ})^{T}_{\beta}(0)$

$(ɗ_{Ⱥ})^{T}_{\beta}((\wp * r) * ɣ) \geq (ɗ_{Ⱥ})^{T}_{\beta}\left((ɰ * (ɰ * ɣ)) * ɣ\right)$

$= (ɗ_{Ⱥ})^{T}_{\beta}((ɰ * ɣ) * (ɰ * ɣ))$ [by P-3]

$= (ɗ_{Ⱥ})^{T}_{\beta}(0)$ [by BCK-3]

so $(ɗ_{Ⱥ})^{T}_{\beta}((\wp * r) * ɣ) = (ɗ_{Ⱥ})^{T}_{\beta}(0)$

and $(Ч_{Ⱥ})^{T}_{\beta}((\wp * r) * ɣ) \leq (Ч_{Ⱥ})^{T}_{\beta}\left((ɰ * (ɰ * ɣ)) * ɣ\right)$

$= (Ч_{Ⱥ})^{T}_{\beta}((ɰ * ɣ) * (ɰ * ɣ))$ [by P-3]

$= (Ч_{Ⱥ})^{T}_{\beta}(0)$ [by BCK-3]

so $(Ч_{Ⱥ})^{T}_{\beta}((\wp * r) * ɣ) = (Ч_{Ⱥ})^{T}_{\beta}(0)$.

By applying conditions (P-3), ($\mathcal{NFPII}$ -2, 3 & 4), we obtain

$(\mathbb{P}_{Ⱥ})^{T}_{\beta}((ѣ * ɣ) * (ɰ * ɣ)) = (\mathbb{P}_{Ⱥ})^{T}_{\beta}\left((ѣ * (ɰ * ɣ)) * ɣ\right)$

$\geq min\{(\mathbb{P}_{Ⱥ})^{T}_{\beta}((\wp * r) * ɣ), (\mathbb{P}_{Ⱥ})^{T}_{\beta}(r * ɣ)\}$

$= min\{(\mathbb{P}_{Ⱥ})^{T}_{\beta}(0), (\mathbb{P}_{Ⱥ})^{T}_{\beta}(r * ɣ)\}$

$= (\mathbb{P}_{Ⱥ})^{T}_{\beta}(r * ɣ) = (\mathbb{P}_{Ⱥ})^{T}_{\beta}((ѣ * ɰ) * ɣ)$

$(ɗ_{Ⱥ})^{T}_{\beta}((ѣ * ɣ) * (ɰ * ɣ)) = (ɗ_{Ⱥ})^{T}_{\beta}\left((ѣ * (ɰ * ɣ)) * ɣ\right)$

$\geq min\{(ɗ_{Ⱥ})^{T}_{\beta}((\wp * r) * ɣ), (ɗ_{Ⱥ})^{T}_{\beta}(r * ɣ)\}$

$= min\{(ɗ_{Ⱥ})^{T}_{\beta}(0), (ɗ_{Ⱥ})^{T}_{\beta}(r * ɣ)\}$

$= (ɗ_{Ⱥ})^{T}_{\beta}(r * ɣ) = (ɗ_{Ⱥ})^{T}_{\beta}((ѣ * ɰ) * ɣ)$ and

$(Ч_{Ⱥ})^{T}_{\beta}((ѣ * ɣ) * (ɰ * ɣ)) = (Ч_{Ⱥ})^{T}_{\beta}\left((ѣ * (ɰ * ɣ)) * ɣ\right)$

$\leq max\{(Ч_{Ⱥ})^{T}_{\beta}((\wp * r) * ɣ), (Ч_{Ⱥ})^{T}_{\beta}(r * ɣ)\}$

$= max\{(Ч_{Ⱥ})^{T}_{\beta}(0), (Ч_{Ⱥ})^{T}_{\beta}(r * ɣ)\}$

$= (Ч_{Ⱥ})^{T}_{\beta}(r * ɣ) = (Ч_{Ⱥ})^{T}_{\beta}((ѣ * ɰ) * ɣ)$.

Thus, $(\mathbb{P}_{Ⱥ})^{T}_{\beta}((ѣ * ɣ) * (ɰ * ɣ)) \geq (\mathbb{P}_{Ⱥ})^{T}_{\beta}((ѣ * ɰ) * ɣ)$

$(ɗ_{Ⱥ})^{T}_{\beta}((ѣ * ɣ) * (ɰ * ɣ)) \geq (ɗ_{Ⱥ})^{T}_{\beta}((ѣ * ɰ) * ɣ)$ and

$(Ч_{Ⱥ})^{T}_{\beta}((ѣ * ɣ) * (ɰ * ɣ)) \leq (Ч_{Ⱥ})^{T}_{\beta}((ѣ * ɰ) * ɣ)\ \forall\, ѣ, ɰ, ɣ \in Ɠ$.

In the converse direction, assume that $Ⱥ^{T}_{\beta}$ is a $\mathcal{NFI}$ of Ɠ satisfies the inequalities

$(\mathbb{P}_{Ⱥ})^{T}_{\beta}((ѣ * ɣ) * (ɰ * ɣ)) \geq (\mathbb{P}_{Ⱥ})^{T}_{\beta}((ѣ * ɰ) * ɣ)$

$(ɗ_{Ⱥ})^{T}_{\beta}((ѣ * ɣ) * (ɰ * ɣ)) \geq (ɗ_{Ⱥ})^{T}_{\beta}((ѣ * ɰ) * ɣ)$ and

$(Ч_{Ⱥ})^{T}_{\beta}((ѣ * ɣ) * (ɰ * ɣ)) \leq (Ч_{Ⱥ})^{T}_{\beta}((ѣ * ɰ) * ɣ)\ \forall\, ѣ, ɰ, ɣ \in Ɠ$.

For any $ѣ, ɰ, ɣ \in Ɠ$, By applying conditions ($\mathcal{NFPII}$ -2, 3 and 4), we obtain

$(\mathbb{P}_{Ⱥ})^{T}_{\beta}(ѣ * ɣ) \geq min\{(\mathbb{P}_{Ⱥ})^{T}_{\beta}((ѣ * ɣ) * (ɰ * ɣ)), (\mathbb{P}_{Ⱥ})^{T}_{\beta}(ɰ * ɣ)\}$

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$$\geq min\{(\mathbb{P}_{Ⱥ})^T_\beta((ѣ * щ) * ɣ), (\mathbb{P}_{Ⱥ})^T_\beta(щ * ɣ)\}$$

$$(ɗ_{Ⱥ})^T_\beta(ѣ * ɣ) \geq min\{(ɗ_{Ⱥ})^T_\beta((ѣ * ɣ) * (щ * ɣ)), (ɗ_{Ⱥ})^T_\beta(щ * ɣ)\}$$

$$\geq min\{(ɗ_{Ⱥ})^T_\beta((ѣ * щ) * ɣ), (ɗ_{Ⱥ})^T_\beta(щ * ɣ)\} \text{ and}$$

$$(Ч_{Ⱥ})^T_\beta(ѣ * ɣ) \leq max\{(Ч_{Ⱥ})^T_\beta((ѣ * ɣ) * (щ * ɣ)), (Ч_{Ⱥ})^T_\beta(щ * ɣ)\}$$

$$\leq max\{(Ч_{Ⱥ})^T_\beta((ѣ * щ) * ɣ), (Ч_{Ⱥ})^T_\beta(щ * ɣ)\} \ \forall\, ѣ, щ, ɣ \in Ɠ.$$

Therefore, $Ⱥ^T_\beta$ is a $\mathcal{NFPII}$ of Ɠ.

Theorem 6.11 Let $Ⱥ^T_\beta$ be a $\mathcal{NFI}$ of Ɠ. If $Ⱥ^T_\beta$ is a $\mathcal{NFPII}$ of Ɠ then the inequalities are satisfied.

(1) $(\mathbb{P}_{Ⱥ})^T_\beta(ѣ * щ) \geq (\mathbb{P}_{Ⱥ})^T_\beta((ѣ * щ) * щ)$

(2) $(ɗ_{Ⱥ})^T_\beta(ѣ * щ) \geq (ɗ_{Ⱥ})^T_\beta((ѣ * щ) * щ)$

(3) $(Ч_{Ⱥ})^T_\beta(ѣ * щ) \leq (Ч_{Ⱥ})^T_\beta((ѣ * щ) * щ) \ \forall\, ѣ, щ \in Ɠ.$

Theorem 6.12. If $Ⱥ^T_\beta$ is a $\mathcal{NFPII}$ of Ɠ, then

(1) For $ѣ, щ, \wp, r \in Ɠ$, $((ѣ * щ) * щ) * \wp \leq r$ implies
$(\mathbb{P}_{Ⱥ})^T_\beta(ѣ * щ) \geq min\{(\mathbb{P}_{Ⱥ})^T_\beta(\wp), (\mathbb{P}_{Ⱥ})^T_\beta(r)\}$, $(ɗ_{Ⱥ})^T_\beta(ѣ * щ) \geq min\{(ɗ_{Ⱥ})^T_\beta(\wp), (ɗ_{Ⱥ})^T_\beta(r)\}$ and
$(Ч_{Ⱥ})^T_\beta(ѣ * щ) \leq max\{(Ч_{Ⱥ})^T_\beta(\wp), (Ч_{Ⱥ})^T_\beta(r)\}$.

(2) For $ѣ, щ, \wp, r \in Ɠ$, $((ѣ * щ) * щ) * \wp \leq r$ implies
$(\mathbb{P}_{Ⱥ})^T_\beta((ѣ * ɣ) * (щ * ɣ)) \geq min\{(\mathbb{P}_{Ⱥ})^T_\beta(\wp), (\mathbb{P}_{Ⱥ})^T_\beta(r)\}$,
$(ɗ_{Ⱥ})^T_\beta((ѣ * ɣ) * (щ * ɣ)) \geq min\{(ɗ_{Ⱥ})^T_\beta(\wp), (ɗ_{Ⱥ})^T_\beta(r)\}$ and
$(Ч_{Ⱥ})^T_\beta((ѣ * ɣ) * (щ * ɣ)) \leq max\{(Ч_{Ⱥ})^T_\beta(\wp), (Ч_{Ⱥ})^T_\beta(r)\}$.

Proof: Let $Ⱥ^T_\beta$ be a $\mathcal{NFPII}$ of Ɠ. And let $ѣ, щ, \wp, r \in Ɠ$ such that $((ѣ * щ) * щ) * \wp \leq r$. We have

$$(\mathbb{P}_{Ⱥ})^T_\beta((ѣ * щ) * щ) \geq min\{(\mathbb{P}_{Ⱥ})^T_\beta(\wp), (\mathbb{P}_{Ⱥ})^T_\beta(r)\}$$

$$(ɗ_{Ⱥ})^T_\beta((ѣ * щ) * щ) \geq min\{(ɗ_{Ⱥ})^T_\beta(\wp), (ɗ_{Ⱥ})^T_\beta(r)\} \text{ and}$$

$$(Ч_{Ⱥ})^T_\beta((ѣ * щ) * щ) \leq max\{(Ч_{Ⱥ})^T_\beta(\wp), (Ч_{Ⱥ})^T_\beta(r)\}.$$

Insert $ɣ = щ$ in $\mathcal{NFPII}$ -2, 3 and 4. We obtain

$$(\mathbb{P}_{Ⱥ})^T_\beta(ѣ * щ) \geq min\{(\mathbb{P}_{Ⱥ})^T_\beta((ѣ * щ) * щ), (\mathbb{P}_{Ⱥ})^T_\beta(щ * щ)\}$$

$$= min\{(\mathbb{P}_{Ⱥ})^T_\beta((ѣ * щ) * щ), (\mathbb{P}_{Ⱥ})^T_\beta(0)\}$$

$$= (\mathbb{P}_{Ⱥ})^T_\beta((ѣ * щ) * щ) \geq min\{(\mathbb{P}_{Ⱥ})^T_\beta(\wp), (\mathbb{P}_{Ⱥ})^T_\beta(r)\}.$$

$$(ɗ_{Ⱥ})^T_\beta(ѣ * щ) \geq min\{(ɗ_{Ⱥ})^T_\beta((ѣ * щ) * щ), (ɗ_{Ⱥ})^T_\beta(щ * щ)\}$$

$$= min\{(ɗ_{Ⱥ})^T_\beta((ѣ * щ) * щ), (ɗ_{Ⱥ})^T_\beta(0)\}$$

$$= (ɗ_{Ⱥ})^T_\beta((ѣ * щ) * щ) \geq min\{(ɗ_{Ⱥ})^T_\beta(\wp), (ɗ_{Ⱥ})^T_\beta(r)\} \text{ and}$$

$$(Ч_{Ⱥ})^T_\beta(ѣ * щ) \leq max\{(Ч_{Ⱥ})^T_\beta((ѣ * щ) * щ), (Ч_{Ⱥ})^T_\beta(щ * щ)\}$$

$$= max\{(Ч_{Ⱥ})^T_\beta((ѣ * щ) * щ), (Ч_{Ⱥ})^T_\beta(0)\}$$

$$= (Ч_{Ⱥ})^T_\beta((ѣ * щ) * щ) \leq max\{(Ч_{Ⱥ})^T_\beta(\wp), (Ч_{Ⱥ})^T_\beta(r)\}.$$

Therefore, $(\mathbb{P}_{Ⱥ})^T_\beta(ѣ * щ) \geq min\{(\mathbb{P}_{Ⱥ})^T_\beta(\wp), (\mathbb{P}_{Ⱥ})^T_\beta(r)\}$
$(ɗ_{Ⱥ})^T_\beta(ѣ * щ) \geq min\{(ɗ_{Ⱥ})^T_\beta(\wp), (ɗ_{Ⱥ})^T_\beta(r)\}$ and
$(Ч_{Ⱥ})^T_\beta(ѣ * щ) \leq max\{(Ч_{Ⱥ})^T_\beta(\wp), (Ч_{Ⱥ})^T_\beta(r)\}$.

2. Let $ѣ, щ, \wp, r \in Ɠ$ such that $((ѣ * щ) * щ) * \wp \leq r$.
Since $Ⱥ^T_\beta$ is a $\mathcal{NFPII}$ of Ɠ, we obtain

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$$(\mathbb{P}_{Ⱥ})_{\beta}^{T}((ѣ * ɣ) * (щ * ɣ)) \geq (\mathbb{P}_{Ⱥ})_{\beta}^{T}((ѣ * щ) * ɣ)$$
$$\geq min\{(\mathbb{P}_{Ⱥ})_{\beta}^{T}(\wp), (\mathbb{P}_{Ⱥ})_{\beta}^{T}(r)\}$$
$$(ɗ_{Ⱥ})_{\beta}^{T}((ѣ * ɣ) * (щ * ɣ)) \geq (ɗ_{Ⱥ})_{\beta}^{T}((ѣ * щ) * ɣ)$$
$$\geq min\{(ɗ_{Ⱥ})_{\beta}^{T}(\wp), (ɗ_{Ⱥ})_{\beta}^{T}(r)\}$$
$$(Ч_{Ⱥ})_{\beta}^{T}((ѣ * ɣ) * (щ * ɣ)) \leq (Ч_{Ⱥ})_{\beta}^{T}((ѣ * щ) * ɣ)$$
$$\leq max\{(Ч_{Ⱥ})_{\beta}^{T}(\wp), (Ч_{Ⱥ})_{\beta}^{T}(r)\}.$$

This completes the proof.

Theorem 6.13. If $Ⱥ_{\beta}^{T}$ is a $\mathcal{NFI}$ of Ɠ with the following conditions:

$$(\mathbb{P}_{Ⱥ})_{\beta}^{T}(ѣ * щ) \geq min\left\{(\mathbb{P}_{Ⱥ})_{\beta}^{T}\left(((ѣ * щ) * щ) * ɣ\right), (\mathbb{P}_{Ⱥ})_{\beta}^{T}(ɣ)\right\}$$
$$(ɗ_{Ⱥ})_{\beta}^{T}(ѣ * щ) \geq min\left\{(ɗ_{Ⱥ})_{\beta}^{T}\left(((ѣ * щ) * щ) * ɣ\right), (ɗ_{Ⱥ})_{\beta}^{T}(ɣ)\right\} \text{ and}$$
$$(Ч_{Ⱥ})_{\beta}^{T}(ѣ * щ) \leq max\left\{(Ч_{Ⱥ})_{\beta}^{T}\left(((ѣ * щ) * щ) * ɣ\right), (Ч_{Ⱥ})_{\beta}^{T}(ɣ)\right\} \ \forall\, ѣ, щ, ɣ \in Ɠ.$$

Then $Ⱥ_{\beta}^{T}$ is a $\mathcal{NFPII}$ of Ɠ.

Proof: Suppose $Ⱥ_{\beta}^{T}$ is a $\mathcal{NFI}$ of Ɠ, satisfying the following conditions.

$$(\mathbb{P}_{Ⱥ})_{\beta}^{T}(ѣ * щ) \geq min\left\{(\mathbb{P}_{Ⱥ})_{\beta}^{T}\left(((ѣ * щ) * щ) * ɣ\right), (\mathbb{P}_{Ⱥ})_{\beta}^{T}(ɣ)\right\}$$
$$(ɗ_{Ⱥ})_{\beta}^{T}(ѣ * щ) \geq min\left\{(ɗ_{Ⱥ})_{\beta}^{T}\left(((ѣ * щ) * щ) * ɣ\right), (ɗ_{Ⱥ})_{\beta}^{T}(ɣ)\right\} \text{ and}$$
$$(Ч_{Ⱥ})_{\beta}^{T}(ѣ * щ) \leq max\left\{(Ч_{Ⱥ})_{\beta}^{T}\left(((ѣ * щ) * щ) * ɣ\right), (Ч_{Ⱥ})_{\beta}^{T}(ɣ)\right\}$$

Applying (P-3) and (P-4), we get

We have $\left((ѣ * ɣ) * ɣ\right) * (щ * ɣ) \leq (ѣ * ɣ) * щ = (ѣ * щ) * ɣ, \ \forall\, ѣ, щ, ɣ \in Ɠ.$

Thus, applying Lemma 6.8, it follows that,

$$(\mathbb{P}_{Ⱥ})_{\beta}^{T}\left(((ѣ * ɣ) * ɣ) * (щ * ɣ)\right) \geq (\mathbb{P}_{Ⱥ})_{\beta}^{T}((ѣ * щ) * ɣ)$$
$$(ɗ_{Ⱥ})_{\beta}^{T}\left(((ѣ * ɣ) * ɣ) * (щ * ɣ)\right) \geq (ɗ_{Ⱥ})_{\beta}^{T}((ѣ * щ) * ɣ) \text{ and}$$
$$(Ч_{Ⱥ})_{\beta}^{T}\left(((ѣ * ɣ) * ɣ) * (щ * ɣ)\right) \leq (Ч_{Ⱥ})_{\beta}^{T}((ѣ * щ) * ɣ).$$

By assumption,

$$(\mathbb{P}_{Ⱥ})_{\beta}^{T}(ѣ * ɣ) \geq min\left\{(\mathbb{P}_{Ⱥ})_{\beta}^{T}\left(((ѣ * ɣ) * ɣ) * (щ * ɣ)\right), (\mathbb{P}_{Ⱥ})_{\beta}^{T}(щ * ɣ)\right\}$$
$$\geq min\{(\mathbb{P}_{Ⱥ})_{\beta}^{T}((ѣ * щ) * ɣ), (\mathbb{P}_{Ⱥ})_{\beta}^{T}(щ * ɣ)\}$$
$$(ɗ_{Ⱥ})_{\beta}^{T}(ѣ * ɣ) \geq min\left\{(ɗ_{Ⱥ})_{\beta}^{T}\left(((ѣ * ɣ) * ɣ) * (щ * ɣ)\right), (ɗ_{Ⱥ})_{\beta}^{T}(щ * ɣ)\right\}$$
$$\geq min\{(ɗ_{Ⱥ})_{\beta}^{T}((ѣ * щ) * ɣ), (ɗ_{Ⱥ})_{\beta}^{T}(щ * ɣ)\} \text{ and}$$
$$(Ч_{Ⱥ})_{\beta}^{T}(ѣ * ɣ) \leq max\left\{(Ч_{Ⱥ})_{\beta}^{T}\left(((ѣ * ɣ) * ɣ) * (щ * ɣ)\right), (Ч_{Ⱥ})_{\beta}^{T}(щ * ɣ)\right\}$$
$$\leq max\{(Ч_{Ⱥ})_{\beta}^{T}((ѣ * щ) * ɣ), (Ч_{Ⱥ})_{\beta}^{T}(щ * ɣ)\} \ \forall\, ѣ, щ, ɣ \in Ɠ.$$

Thus, we conclude that $Ⱥ_{\beta}^{T}$ is a $\mathcal{NFPII}$ of Ɠ.

VII. Conclusion

This research delves into the application of Left-Right Derivation ((L, R)-D) and Right-Left Derivation ((R, L)-D) a particular derivative approach to develop a deeper understanding (NFSA, NFI, and DNFI). We introduce four new concepts: Derivations of Neutrosophic fuzzy sub-algebra (DNFSA), Derivations of Neutrosophic fuzzy ideal (DNFI), Derivations of Neutrosophic fuzzy implicative ideal (DNFII), and Derivations of Neutrosophic fuzzy positive implicative ideal (DNFPII). And also explore the interrelationships between these concepts, uncover specific outcomes, and investigate various associated properties, ultimately testing a range of related residency outcomes. Finally, we discussed Neutrosophic fuzzy translation to Neutrosophic fuzzy positive implicative ideals in BCK-algebras.

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