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ON INTERVAL-VALUED NEUTROSOPHIC FUZZY HYPER BCK-IDEALS AND IMPLICATIVE HYPER BCK-IDEALS OF HYPER BCK-ALGEBRAS

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ABSTRACT
This study explores the application of interval-valued Neutrosophic fuzzy sets (IVNFS) to hyper BCK-ideals (h-BCK-Is) within hyper BCK-algebras (h-BCK-A's). We introduce the concept of interval-valued Neutrosophic fuzzification (IVN- fuzzification) of (*strong*, *weak*, s-*weak*) h-BCK-Is and establish that every IVNF-s-*weak*-h-BCK-I of K is an IVNF-*weak*-h-BCK-I. Furthermore, we define the notions of Neutrosophic fuzzy (*weak*) implicative hyper BCK-ideals of hyper BCK-algebras and present theorems that characterize these notions according to the level subsets. We also analyze the properties and provide characterizations of IVNF h-BCK-Is, and obtain the relationship among these notions, Neutrosophic fuzzy (*strong*, *weak*, *reflexive*) hyper BCK-ideals, and Neutrosophic fuzzy positive implicative hyper BCK-ideals of types-1, 2 ...8, yielding related results that contribute to the development of Neutrosophic fuzzy set theory in hyper BCK-algebras.

Keywords:
Interval-Valued Neutrosophic Fuzzy-$strong$-hyper BCK-ideal, Interval-Valued Neutrosophic Fuzzy-$weak$-hyper BCK-ideal, Interval-Valued Neutrosophic Fuzzy-s-$weak$-hyper-BCK-ideal, Neutrosophic Fuzzy-($weak$)-Implicative hyper BCK-ideal, Neutrosophic Fuzzy Sets, Hyper BCK-algebras.

I. Introduction

Mathematics is built upon algebraic structures, which have far-reaching applications in various fields, including theoretical physics, computer science, and information science. However, the complexities of uncertainty necessitate the use of non-classical logic (a major development and broadening of classical logic), a more comprehensive and powerful framework than classical logic. Consequently, non-classical logic has emerged as a valuable tool in computer science. Furthermore, non-classical logic is particularly well-suited to handle fuzzy information and uncertainty. Zadeh's groundbreaking work in 1965 [12, 13] pioneered the concept of fuzzy subsets, facilitating the representation of uncertainty in real-world physical systems. Building upon the notion of fuzzy sets (FSs), numerous researchers have expanded the field by developing higher-order fuzzy sets, including interval-valued fuzzy sets (IVFSs) and intuitionistic fuzzy sets (IFSs). The developed frameworks facilitate the effective handling of imperfect and imprecise information. Furthermore, Atanassov's [1, 2] IFSs and the IVIFSs, generalize ordinary fuzzy sets. In 1934, Marty [8] introduced the hyper-structure theory, also known as multi-algebras, at the 8th Congress of Scandinavian Mathematicians, laying the groundwork for subsequent applications. Hyper structures have a wide range of applications in various disciplines of both pure science and applied science. The application of hyper-structure to BCK-algebras by Jun [6] et al. has led to the development of hyper BCK-algebras, which represents a significant generalization of BCK-algebras. Expanding on Atanassov's work, [4] Borzooei and Jun introduced intuitionistic fuzzy (IF) versions of strong, weak, and s-weak hyper BCK-ideals in hyper BCK-algebras, and studied their behaviour. In [7], Jun et al. introduced the notion of implicative hyper BCK-ideals and obtain some related results. Recently, Satyanarayana et.al., [9, 11] introduced the notion of IVIF-h-BCK-Is of hyper BCK-algebras, and also introduced

IFI-h-BCK-Is. Now, in this work we generalized to interval-valued Neutrosophic fuzzy hyper BCK-logic within the hyper BCK-algebras.

We introduce and develops interval-valued Neutrosophic fuzzy (IVNF) hyper BCK-ideals in K, investigating their properties and characteristics. The notions of interval-valued Neutrosophic fuzzy implicative hyper BCK-ideals are defined and examined, revealing relationships with related concepts, including Neutrosophic fuzzy hyper BCK-ideals and this research defines. Additionally, Neutrosophic fuzzy positive implicative hyper BCK-ideals of types 1-8., Furthermore, The relationships among these notions, Neutrosophic fuzzy strong, weak, s-weak, and reflexive hyper BCK-ideals, are also explored, providing related results.
The following abbreviations are utilized throughout this paper:

- h-BCK-A's (or) K : hyper BCK-algebras.
- h-BCK-Is: hyper BCK-ideals.
- FS: fuzzy set.
- IVFS: interval-valued fuzzy set.
- IVIFS: interval-valued intuitionistic fuzzy-set.
- IVNFS: interval-valued Neutrosophic fuzzy set.
- IVIF-h-BCK-Is: interval-valued intuitionistic fuzzy-hyper-BCK-ideals.
- IVIF-(*strong*, *weak*, s-*weak*)-h-BCK-I: interval-valued intuitionistic fuzzy-(*strong*, *weak*, s-*weak*)-hyper BCK-ideal.
- IVNF-h-BCK-Is: interval-valued Neutrosophic fuzzy hyper BCK-ideals.
- IVNF-(*strong*, *weak*, s-*weak*)-h-BCK-I: interval-valued Neutrosophic fuzzy (*strong*, *weak*, s-*weak*) hyper BCK-ideal.
- IVIF(rep., I) PI-h-BCK-I: interval-valued intuitionistic fuzzy (rep., implicative) positive implicative hyper BCK-ideal.
- IVN(rep., I)PI-h-BCK-Is: interval-valued Neutrosophic(rep., implicative) positive implicative hyper BCK-ideal.
- FPII: Fuzzy positive implicative ideal.
- (rep., I).PI-h-BCK-Is: (rep., implicative) positive implicative hyper BCK-ideals.

II. Preliminaries

This section provides, some basic information's about in the present research work, which are crucial for the subsequent development of this article.
Consider a nonempty set K endowed with a hyper operation, denoted by " $\star$ ", which maps $K \times K$ to $\mathcal{Q}^*(K)$, the set of all nonempty subsets of K. For any two subsets $\mathfrak{C}$ & $\mathcal{G}$ of K, the hyper operation is defined as: as $\mathfrak{C} \star \mathcal{G} = \cup_{\mathfrak{m}\in\mathfrak{C},\mathfrak{n}\in\mathcal{G}} \mathfrak{m} \star \mathfrak{n}$. For notational simplicity, we will use $\mathfrak{f} \star \mathfrak{g}$ to represent $\mathfrak{f} \star \{\mathfrak{g}\}$, $\{\mathfrak{f}\} \star \mathfrak{g}$ or $\{\mathfrak{f}\} \star \{\mathfrak{g}\}$.
A h-BCK-A $(K, \star, 0)$ is defined as a nonempty set K equipped with a hyper operation "$\star$" and a constant 0, fulfilling the below conditions:
$(\boldsymbol{hBCK1})$ $(\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \hbar) \ll \mathfrak{f} \star \mathfrak{g}$,
$(\boldsymbol{hBCK2})$ $(\mathfrak{f} \star \mathfrak{g}) \star \hbar = (\mathfrak{f} \star \hbar) \star \mathfrak{g}$,
$(\boldsymbol{hBCK3})$ $\mathfrak{f} \star K \ll \{\mathfrak{f}\}$,
$(\boldsymbol{hBCK4})$ $\mathfrak{f} \ll \mathfrak{g}$ and $\mathfrak{g} \ll \mathfrak{f}$ implies $\mathfrak{f} = \mathfrak{g}$, for all $\mathfrak{f}, \mathfrak{g}, \hbar \in K$.

We can define a relation "$\ll$" on K by letting $\mathfrak{f} \ll \mathfrak{g} \Leftrightarrow 0 \in \mathfrak{f} \star \mathfrak{g}$ and for every $\mathfrak{C}, \mathcal{G} \subseteq K$, $\mathfrak{C} \ll \mathcal{G}$ is defined $\forall\, \mathfrak{m} \in \mathfrak{C}\ \exists\ \mathfrak{n} \in \mathcal{G} \ni \mathfrak{m} \ll \mathfrak{n}$. In such case, we call the relation "$\ll$" the hyper-order in K.
Observe that condition $(\boldsymbol{hBCK3})$ is equivalent to $(\boldsymbol{\mathcal{P}_1})$ $\mathfrak{f} \star \mathfrak{g} \ll \{\mathfrak{f}\}$, for all $\mathfrak{f}, \mathfrak{g} \in K$
The following hold in any h-BCK-A K:
$(\boldsymbol{\mathcal{P}_2})$ $\mathfrak{f} \star 0 \ll \{\mathfrak{f}\}$, $0 \star \mathfrak{f} = \{0\}$ and $0 \star 0 = \{0\}$,

$(\mathcal{P}_3)$ $(\mathfrak{C} \star \mathcal{G}) \star \mathcal{S} = (\mathfrak{C} \star \mathcal{S}) \star \mathcal{G}, \mathfrak{C} \star \mathcal{G} \ll \mathfrak{C}$ and $0 \star \mathfrak{C} = \{0\}$,
$(\mathcal{P}_4)$ $0 \star 0 = \{0\}$,
$(\mathcal{P}_5)$ $0 \ll \mathfrak{f}$,
$(\mathcal{P}_6)$ $\mathfrak{f} \ll \mathfrak{f}$,
$(\mathcal{P}_7)$ $\mathfrak{C} \ll \mathfrak{C}$,
$(\mathcal{P}_8)$ $\mathfrak{C} \subseteq \mathcal{G} \Rightarrow \mathfrak{C} \ll \mathcal{G}$,
$(\mathcal{P}_9)$ $0 \star \mathfrak{f} = \{0\}$, $(\mathcal{P}_{10})$ $\mathfrak{f} \star 0 = \{\mathfrak{f}\}$,
$(\mathcal{P}_{11})$ $0 \star \mathfrak{C} = \{0\}$,
$(\mathcal{P}_{12})$ $\mathfrak{C} \ll \{0\} \Rightarrow \mathfrak{C} = \{0\}$,
$(\mathcal{P}_{13})$ $\mathfrak{C} \star \mathcal{G} \ll \mathfrak{C}$,
$(\mathcal{P}_{14})$ $\mathfrak{f} \in \mathfrak{f} \star 0$,
$(\mathcal{P}_{15})$ $\mathfrak{f} \star 0 \ll \{\mathfrak{g}\} \Rightarrow \mathfrak{f} \ll \mathfrak{g}$,
$(\mathcal{P}_{16})$ $\mathfrak{g} \ll \mathscr{h} \Rightarrow \mathfrak{f} \star \mathscr{h} \ll \mathfrak{f} \star \mathfrak{g}$,
$(\mathcal{P}_{17})$ $\mathfrak{f} \star \mathfrak{g} = \{0\} \Rightarrow (\mathfrak{f} \star \mathscr{h}) \star (\mathfrak{g} \star \mathscr{h}) = \{0\}$ and $\mathfrak{f} \star \mathscr{h} \ll \mathfrak{g} \star \mathscr{h}$,
$(\mathcal{P}_{18})$ $\mathfrak{C} \star \{0\} = \{0\} \Rightarrow \mathfrak{C} = \{0\}$ for all $\mathfrak{f}, \mathfrak{g}, \mathscr{h} \in K$ and for any non-empty sub-sets $\mathfrak{C}$, $\mathcal{G}$ and $\mathcal{S}$ of K.
Let $\mathfrak{I}$ be a non-empty sub-set of h-BCK-A K and $0 \in \mathfrak{I}$. Then $\mathfrak{I}$ is said to be
$(\mathfrak{I}_1)$ a *weak*-h-BCK-I of K, if $\mathfrak{f} \star \mathfrak{g} \subseteq \mathfrak{I}$ and $\mathfrak{g} \in \mathfrak{I} \Rightarrow \mathfrak{f} \in \mathfrak{I}, \forall\, \mathfrak{f}, \mathfrak{g} \in K$.
$(\mathfrak{I}_2)$ a h-BCK-I of K, if $\mathfrak{f} \star \mathfrak{g} \ll \mathfrak{I}$ and $\mathfrak{g} \in \mathfrak{I} \Rightarrow \mathfrak{f} \in \mathfrak{I}, \forall\, \mathfrak{f}, \mathfrak{g} \in K$.
$(\mathfrak{I}_3)$ a *strong* -h-BCK-I of K, if $\mathfrak{f} \star \mathfrak{g} \cap \mathfrak{I} \neq \emptyset$ and $\mathfrak{g} \in \mathfrak{I} \Rightarrow \mathfrak{f} \in \mathfrak{I}, \forall\, \mathfrak{f}, \mathfrak{g} \in K$.
$(\mathfrak{I}_4)$ $\mathfrak{I}$ is said to be *reflexive* if $\mathfrak{f} \star \mathfrak{f} \subseteq \mathfrak{I}, \forall\, \mathfrak{f} \in K$.
$(\mathfrak{I}_5)$ S-*reflexive*, if $\star\, \mathfrak{g} \cap \mathfrak{I} \neq \emptyset \Rightarrow \mathfrak{f} \star \mathfrak{g} \ll \mathfrak{I}, \forall\, \mathfrak{f}, \mathfrak{g} \in K$.
$(\mathfrak{I}_6)$ closed if $\mathfrak{f} \ll \mathfrak{I}$ and $\mathfrak{g} \in \mathfrak{I} \Rightarrow \mathfrak{f} \in \mathfrak{I}, \forall\, \mathfrak{f}, \mathfrak{g} \in K$.
Every S-*reflexive* subset of K is clearly reflexive.

Let K be a h-BCK-A then K is said to be a PI-h-BCK-A, if for all $\mathfrak{f}, \mathfrak{g}, \mathscr{h} \in K$, $(\mathfrak{f} \star \mathfrak{g}) \star \mathscr{h} = (\mathfrak{f} \star \mathscr{h}) \star (\mathfrak{g} \star \mathscr{h})$ [5].

Let $\mathfrak{I}$ be a nonempty subset of K and $0 \in \mathfrak{I}$. Then $\mathfrak{I}$ is called to be a (*weak*-I-h-BCK-I) *weak* implicative hyper BCK-ideal of K if $(\mathfrak{f} \star \mathscr{h}) \star (\mathfrak{g} \star \mathfrak{f}) \subseteq \mathfrak{I}$ and $\mathscr{h} \in \mathfrak{I} \Rightarrow \mathfrak{f} \in \mathfrak{I}$ an implicative hyper BCK-ideal of K, if $(\mathfrak{f} \star \mathscr{h}) \star (\mathfrak{g} \star \mathfrak{f}) \ll \mathfrak{I}$ and $\mathscr{h} \in \mathfrak{I} \Rightarrow \mathfrak{f} \in \mathfrak{I}, \forall\, \mathfrak{f}, \mathfrak{g}, \mathscr{h} \in K$.

A fuzzy set(FS) in a set K is a function $\xi: K \to [0,1]$, and the complement of ξ, denoted by ξ^c, is the FS in K given by $\xi^c(\mathfrak{f}) = 1 - \xi(\mathfrak{f})$, for all $\mathfrak{f} \in K$. Let ξ and ϖ be the FSs of K. For $\mathscr{s}, \mathscr{v} \in [0,1]$ the set $\mathcal{U}(\xi; \mathscr{s}) = \{\mathfrak{f} \in \mathfrak{A} \mid \xi(\mathfrak{f}) \geq \mathscr{s}\}$ is called upper $\mathscr{s}$-level cut of ξ and the set $\mathcal{L}(\varpi; \mathscr{v}) = \{\mathfrak{f} \in \mathfrak{A} \mid \varpi(\mathfrak{f}) \leq \mathscr{v}\}$ is called lower $\mathscr{v}$-level cut of ϖ.

Let ξ be a fuzzy sub-set of K and $\xi(0) \geq \xi(\mathfrak{f}), \forall\, \mathfrak{f} \in K$. Then ξ is said to be a

i. (F-*weak*-I-h-BCK-I) fuzzy *weak* implicative hyper BCK-ideal of K if

$$\xi(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{m} \in (\mathfrak{f} \star \mathscr{h}) \star (\mathfrak{g} \star \mathfrak{f})} \xi(\mathfrak{m}), \xi(\mathscr{h})\right\},$$

ii. (FI-h-BCK-I) fuzzy implicative hyper BCK-ideal of K, if

$$\xi(\mathfrak{f}) \geq \min\left\{\sup_{\mathfrak{m} \in (\mathfrak{f} \star \mathscr{h}) \star (\mathfrak{g} \star \mathfrak{f})} \xi(\mathfrak{m}), \xi(\mathscr{h})\right\},$$

$\forall\, \mathfrak{f}, \mathfrak{g}, \mathscr{h} \in K$.
Although comparing two real numbers to determine the max and min is a simple task, extending this comparison to intervals is more complex. Biswas [3] introduced a methodology to calculate the max/sup and min/inf of two intervals and interval collections.
In the context of interval numbers $\tilde{\mathscr{p}}$ on $[0,1]$, we mean (cf.[2]) an interval $[\mathscr{p}^-, \mathscr{p}^+]$,

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where $0 \le \wp^- \le \wp^+ \le 1$. We denote by $\mathbb{A}[0,1]$ the collection of all closed sub-intervals of the interval $[0,1]$. The interval $[\wp, \wp]$ is identified with the number $\wp \in [0,1]$.

For an interval numbers $\tilde{\wp}_i = [\wp_i^-,\ \mathfrak{q}_i^+] \in \mathbb{A}[0,1]$, $i \in \Im$.
We define

$$\inf \tilde{\wp}_i = \left[\min_{i\in\Im} \wp_i^-, \min_{i\in\Im} \mathfrak{q}_i^+\right]$$

$$\sup \tilde{\wp}_i = \left[\max_{i\in\Im} \wp_i^-, \max_{i\in\Im} \mathfrak{q}_i^+\right]$$

And put

(i) $\tilde{\wp}_1 \cap \tilde{\wp}_2 = \min(\tilde{\wp}_1, \tilde{\wp}_2) = \min([\wp_1^-,\ \mathfrak{q}_1^+], [\wp_2^-,\ \mathfrak{q}_2^+]) = [\min\{\wp_1^-, \wp_2^+\}, \min\{\mathfrak{q}_1^-, \mathfrak{q}_2^+\}]$

(ii) $\tilde{\wp}_1 \cup \tilde{\wp}_2 = \max(\tilde{\wp}_1, \tilde{\wp}_2) = \max([\wp_1^-,\ \mathfrak{q}_1^+], [\wp_2^-,\ \mathfrak{q}_2^+]) = [\max\{\wp_1^-, \wp_2^+\}, \max\{\mathfrak{q}_1^-, \mathfrak{q}_2^+\}]$

(iii) $\tilde{\wp}_1 + \tilde{\wp}_2 = [\wp_1^- + \wp_2^- - \wp_1^-.\wp_2^-,\ \mathfrak{q}_1^+ + \mathfrak{q}_2^+ - \mathfrak{q}_1^+.\mathfrak{q}_2^+]$

(iv) $\tilde{\wp}_1 \le \tilde{\wp}_2 \Leftrightarrow \wp_1^- \le \wp_2^-$ and $\mathfrak{q}_1^+ \le \mathfrak{q}_2^+$

(v) $\tilde{\wp}_1 = \tilde{\wp}_2 \Leftrightarrow \wp_1^- = \wp_2^-$ and $\mathfrak{q}_1^+ = \mathfrak{q}_2^+$,

(vi) $m\mathbb{A} = m[\wp_1^-,\ \mathfrak{q}_1^+] = [m\wp_1^-,\ m\mathfrak{q}_1^+]$, where $0 \le m \le 1$.

It is evident that, the structure $(\mathbb{A}[0,1],\ \le,\ \vee,\ \wedge)$ constitutes a complete lattice with $[0,0]$ and $[1,1]$ serving as its least and greatest elements, respectively.

Assigning membership values has proven to be a challenging task for decision makers. To address this issue, Zadeh [12] introduced s IVFSs, where membership values are represented as intervals within $[0,1]$, rather than single numerical values. We denote the collection of all closed subintervals of $[0,1]$ as $\mathbb{A}[0,1]$.

Let K be a given non-empty set. An IVFS " $\widetilde{\mathfrak{N}}$ " over K is an object having the form $\widetilde{\mathfrak{N}} = \{(\mathfrak{f}, [\xi_{\mathfrak{N}}^-(\mathfrak{f}), \xi_{\mathfrak{N}}^+(\mathfrak{f})]) : \mathfrak{f} \in K\}$, where $\xi_{\mathfrak{N}}^-(\mathfrak{f})$ and $\xi_{\mathfrak{N}}^+(\mathfrak{f})$ are FSs of K such that $\xi_{\mathfrak{N}}^-(\mathfrak{f}) \le \xi_{\mathfrak{N}}^+(\mathfrak{f})$ for all $\mathfrak{f} \in K$. Let $\tilde{\xi}_{\mathfrak{N}}(\mathfrak{f}) = [\xi_{\mathfrak{N}}^-(\mathfrak{f}), \xi_{\mathfrak{N}}^+(\mathfrak{f})]$ then $\widetilde{\mathfrak{N}} = \left\{\left(\mathfrak{f}, \tilde{\xi}_{\mathfrak{N}}(\mathfrak{f})\right) : \mathfrak{f} \in K\right\}$, where $\tilde{\xi}_{\mathfrak{N}} : K \to \mathbb{A}[0,1]$.

Building on the foundations of IFS and IVFS, Atanassov and Gargav [2] introduced IVIFSs, a generalized framework encompassing both IFS and IVFS concepts.

An IVIFS " $\widetilde{\mathfrak{K}}$ " over K is an object having the form $\widetilde{\mathfrak{K}} = \left\{\left(\mathfrak{f}, \tilde{\xi}_{\mathfrak{K}}(\mathfrak{f}), \tilde{\varpi}_{\mathfrak{K}}(\mathfrak{f})\right) : \mathfrak{f} \in K\right\}$, where $\tilde{\xi}_{\mathfrak{K}} : K \to \mathbb{A}[0,1]$, and $\tilde{\varpi}_{\mathfrak{K}} : K \to \mathbb{A}[0,1]$, the intervals $\tilde{\xi}_{\mathfrak{K}}(\mathfrak{f})$ and $\tilde{\varpi}_{\mathfrak{K}}(\mathfrak{f})$ represent the degree of membership and non-membership, respectively, of element $\mathfrak{f}$ to the set $\widetilde{\mathfrak{K}}$, where $\tilde{\xi}_{\mathfrak{K}}(\mathfrak{f}) = [\xi_{\mathfrak{K}}^-(\mathfrak{f}), \xi_{\mathfrak{K}}^+(\mathfrak{f})]$, and $\tilde{\varpi}_{\mathfrak{K}}(\mathfrak{f}) = [\varpi_{\mathfrak{K}}^-(\mathfrak{f}), \varpi_{\mathfrak{K}}^+(\mathfrak{f})]$ for all $\mathfrak{f} \in \mathfrak{A}$ with the condition $[0,0] \le \tilde{\xi}_{\mathfrak{K}}(\mathfrak{f}) + \tilde{\varpi}_{\mathfrak{K}}(\mathfrak{f}) \le [1,1]$ for all $\mathfrak{f} \in K$.

An IVNFS " $\widetilde{\mathfrak{M}}$ " over K is an object having the form $\widetilde{\mathfrak{M}} = \left\{\left(\mathfrak{f}, \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f})\right) : \mathfrak{f} \in K\right\}$, where $\tilde{\xi}_{\mathfrak{M}} : K \to \mathbb{A}[0,1]$, $\tilde{\zeta}_{\mathfrak{M}} : K \to \mathbb{A}[0,1]$ and $\tilde{\varpi}_{\mathfrak{M}} : K \to \mathbb{A}[0,1]$, the intervals $\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f})$ and $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f})$ represent the degree of membership, indeterminacy and non-membership, respectively, of the element $\mathfrak{f}$ to the set $\widetilde{\mathfrak{M}}$, where $\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) = [\xi_{\mathfrak{M}}^-(\mathfrak{f}), \xi_{\mathfrak{M}}^+(\mathfrak{f})]$, $\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) = [\zeta_{\mathfrak{M}}^-(\mathfrak{f}), \zeta_{\mathfrak{M}}^+(\mathfrak{f})]$ and $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) = [\varpi_{\mathfrak{M}}^-(\mathfrak{f}), \varpi_{\mathfrak{M}}^+(\mathfrak{f})]$ for all $\mathfrak{f} \in \mathfrak{A}$ with the condition $[0,0] \le \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) + \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) + \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \le [1,1]$ for all

$\mathfrak{f} \in \mathfrak{A}$. For the purpose of clarity, we introduce the notation $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}})$ is represented IVNFS, where $\mathbb{A}[0,1]$ is the collection of all closed sub-intervals of the interval $[0,1]$.
Let $\tilde{\xi}$ be an interval-valued fuzzy sub-set of K and $\tilde{\xi}(0) \geq \tilde{\xi}(\mathfrak{f}), \forall \mathfrak{f} \in K$. Then $\tilde{\xi}$ is said to be a **(IVF-*weak*-I-h-BCK-I-1)** interval-valued fuzzy *weak* implicative hyper BCK-ideal of K if

$$\tilde{\xi}(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{m}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\xi}(\mathfrak{m}), \tilde{\xi}(\hbar)\right\},$$

(IVFI-h-BCK-I-2) interval-valued fuzzy implicative hyper BCK-ideal of K, if

$$\tilde{\xi}(\mathfrak{f}) \geq \min\left\{\sup_{\mathfrak{m}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\xi}(\mathfrak{m}), \tilde{\xi}(\hbar)\right\}, \quad \forall\, \mathfrak{f}, \mathfrak{g}, \hbar \in K.$$

Definition 2.1[9]. An IVIFS $\widetilde{\mathfrak{K}} = (\tilde{\xi}_{\mathfrak{K}}, \widetilde{\varpi}_{\mathfrak{K}})$ in K is said to be an IVIF-h-BCK-I of K if it fulfils
$(\boldsymbol{IVIFhBCKI-1})$ $\mathfrak{f} \ll \mathfrak{g} \Rightarrow \tilde{\xi}_{\mathfrak{K}}(\mathfrak{f}) \geq \tilde{\xi}_{\mathfrak{K}}(\mathfrak{g})$, and $\widetilde{\varpi}_{\mathfrak{K}}(\mathfrak{f}) \leq \widetilde{\varpi}_{\mathfrak{K}}(\mathfrak{g})$
$(\boldsymbol{IVIFhBCKI-2})$ $\tilde{\xi}_{\mathfrak{K}}(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{m}\in\mathfrak{f}\star\mathfrak{g}} \tilde{\xi}_{\mathfrak{K}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{K}}(\mathfrak{g})\right\}$
$(\boldsymbol{IVIFhBCKI-3})$ $\widetilde{\varpi}_{\mathfrak{K}}(\mathfrak{f}) \leq \max\left\{\sup_{\mathfrak{n}\in\mathfrak{f}\star\mathfrak{g}} \widetilde{\varpi}_{\mathfrak{K}}(\mathfrak{n}), \widetilde{\varpi}_{\mathfrak{K}}(\mathfrak{g})\right\}$, for all $\mathfrak{f}, \mathfrak{g} \in K$.

Definition 2.2[9]. An IVIFS $\widetilde{\mathfrak{K}} = (\tilde{\xi}_{\mathfrak{K}}, \widetilde{\varpi}_{\mathfrak{K}})$ in K is said to be an IVIF-*strong*-h-BCK-I of K if it fulfils
$(\boldsymbol{IVIFShBCKI-1})$ $\inf_{\mathfrak{m}\in\mathfrak{f}\star\mathfrak{g}} \tilde{\xi}_{\mathfrak{K}}(\mathfrak{m}) \geq \tilde{\xi}_{\mathfrak{K}}(\mathfrak{f}) \geq \min\left\{\sup_{\mathfrak{n}\in\mathfrak{f}\star\mathfrak{g}} \tilde{\xi}_{\mathfrak{K}}(\mathfrak{n}), \tilde{\xi}_{\mathfrak{K}}(\mathfrak{g})\right\}$ and
$(\boldsymbol{IVIFShBCKI-2})$ $\sup_{\mathfrak{x}\in\mathfrak{f}\star\mathfrak{g}} \widetilde{\varpi}_{\mathfrak{K}}(\mathfrak{x}) \leq \widetilde{\varpi}_{\mathfrak{K}}(\mathfrak{f}) \leq \max\left\{\inf_{\mathfrak{y}\in\mathfrak{f}\star\mathfrak{g}} \widetilde{\varpi}_{\mathfrak{K}}(\mathfrak{y}), \widetilde{\varpi}_{\mathfrak{K}}(\mathfrak{g})\right\}$,
for all $\mathfrak{f}, \mathfrak{g} \in K$.

Definition 2.3[9]. An IVIF $\widetilde{\mathfrak{K}} = (\tilde{\xi}_{\mathfrak{K}}, \widetilde{\varpi}_{\mathfrak{K}})$ in K is said to be an IVIF-s-*weak*-h-BCK-I of K if it fulfils
$(IVIFsWhBCKI-1)$ $\tilde{\xi}_{\mathfrak{K}}(0) \geq \tilde{\xi}_{\mathfrak{K}}(\mathfrak{g})$, and $\widetilde{\varpi}_{\mathfrak{K}}(0) \leq \widetilde{\varpi}_{\mathfrak{K}}(\mathfrak{g})$, for all $\mathfrak{f}, \mathfrak{g} \in K$.
$(IVIFsWhBCKI-2)$ for every $\mathfrak{f}, \mathfrak{g} \in K$ there exists $\mathfrak{m}, \mathfrak{n} \in \mathfrak{f} \star \mathfrak{g}$ such that

$$\tilde{\xi}_{\mathfrak{K}}(\mathfrak{f}) \geq \min\{\tilde{\xi}_{\mathfrak{K}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{K}}(\mathfrak{g})\} \text{ and } \widetilde{\varpi}_{\mathfrak{K}}(\mathfrak{f}) \leq \max\{\widetilde{\varpi}_{\mathfrak{K}}(\mathfrak{n}), \widetilde{\varpi}_{\mathfrak{K}}(\mathfrak{g})\}.$$

Definition 2.4[9]. An IVIF $\widetilde{\mathfrak{K}} = (\tilde{\xi}_{\mathfrak{K}}, \widetilde{\varpi}_{\mathfrak{K}})$ in K is said to be an IVIF-*weak*-h-BCK-I of K if it fulfils
$(\boldsymbol{IVIFWhBCKI-1})$ $\tilde{\xi}_{\mathfrak{K}}(0) \geq \tilde{\xi}_{\mathfrak{K}}(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{m}\in\mathfrak{f}\star\mathfrak{g}} \tilde{\xi}_{\mathfrak{K}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{K}}(\mathfrak{g})\right\}$ and
$(\boldsymbol{IVIFWhBCKI-2})$ $\widetilde{\varpi}_{\mathfrak{K}}(0) \leq \widetilde{\varpi}_{\mathfrak{K}}(\mathfrak{f}) \leq \max\left\{\sup_{\mathfrak{n}\in\mathfrak{f}\star\mathfrak{g}} \widetilde{\varpi}_{\mathfrak{K}}(\mathfrak{n}), \widetilde{\varpi}_{\mathfrak{K}}(\mathfrak{g})\right\}$,
for all $\mathfrak{f}, \mathfrak{g} \in K$.

Definition 2.5[9]. An IVIFS $\widetilde{\mathfrak{K}} = (\tilde{\xi}_{\mathfrak{K}}, \widetilde{\varpi}_{\mathfrak{K}})$ in K is said to fulfil the "inf-sup" property if for any sub-set $\mathcal{Z}$ of $K \ni \mathfrak{f}_0, \mathfrak{g}_0 \in \mathcal{Z} \ni \tilde{\xi}_{\mathfrak{K}}(\mathfrak{f}_0) = \inf_{\mathfrak{f}\in\mathcal{Z}} \tilde{\xi}_{\mathfrak{K}}(\mathfrak{f})$ and $\widetilde{\varpi}_{\mathfrak{K}}(\mathfrak{g}_0) = \sup_{\mathfrak{g}\in\mathcal{Z}} \widetilde{\varpi}_{\mathfrak{K}}(\mathfrak{g})$.

An IVIFS $\widetilde{\mathfrak{K}} = (\tilde{\xi}_{\mathfrak{K}}, \widetilde{\varpi}_{\mathfrak{K}})$ in K is said to fulfil the "sup-inf" property if for any sub-set $\mathcal{Z}$ of K there exists $\mathfrak{f}_0, \mathfrak{g}_0 \in \mathcal{Z}$ such that

$$\tilde{\xi}_{\mathfrak{K}}(\mathfrak{f}_0) = \sup_{\mathfrak{f}\in\mathcal{Z}} \tilde{\xi}_{\mathfrak{K}}(\mathfrak{f}) \ \textit{and}\ \widetilde{\varpi}_{\mathfrak{K}}(\mathfrak{g}_0) = \inf_{\mathfrak{g}\in\mathcal{Z}} \widetilde{\varpi}_{\mathfrak{K}}(\mathfrak{g}).$$

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Definition 2.6. Let $\widetilde{\mathfrak{K}} = (\tilde{\xi}_{\mathfrak{K}}, \tilde{\varpi}_{\mathfrak{K}})$ be an IVIFS on K and $\tilde{\xi}_{\mathfrak{K}}(0) \geq \tilde{\xi}_{\mathfrak{K}}(\mathfrak{f})$, $\tilde{\varpi}_{\mathfrak{K}}(0) \leq \tilde{\varpi}_{\mathfrak{K}}(\mathfrak{f})$, $\forall\, \mathfrak{f}, \mathfrak{g} \in K$. Then $\widetilde{\mathfrak{M}}$ is said to be an

(i) IVIF-$weak$-I-h-BCK-I of K, if

$$\tilde{\xi}_{\mathfrak{K}}(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{m}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\xi}_{\mathfrak{K}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{K}}(\hbar)\right\}$$

$$\tilde{\varpi}_{\mathfrak{K}}(\mathfrak{f}) \leq \max\left\{\sup_{\mathfrak{x}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\varpi}_{\mathfrak{K}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{K}}(\hbar)\right\}$$

(ii) IVIF-I-h-BCK-I of K, if

$$\tilde{\xi}_{\mathfrak{K}}(\mathfrak{f}) \geq \min\left\{\sup_{\mathfrak{m}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\xi}_{\mathfrak{K}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{K}}(\hbar)\right\},$$

$$\tilde{\varpi}_{\mathfrak{K}}(\mathfrak{f}) \leq \max\left\{\inf_{\mathfrak{x}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\varpi}_{\mathfrak{K}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{K}}(\hbar)\right\}$$

$\forall\, \mathfrak{f}, \mathfrak{g}, \hbar \in K$.

Definition 2.7[10]. Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ be an IVIF subset of K and $\tilde{\xi}_{\mathfrak{M}}(0) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f})$, $\tilde{\varpi}_{\mathfrak{M}}(0) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f})$, $\forall\, \mathfrak{f}, \mathfrak{g} \in K$. Then $\widetilde{\mathfrak{M}}$ is said to be an IVIFPI-h-BCK-I of:

($\boldsymbol{IVNFPIhBCKI_1}$) Type 1, if for all $\mathcal{v} \in \mathfrak{f} \star \hbar$,

$$\tilde{\xi}_{\mathfrak{M}}(\mathcal{v}) \geq \min\left\{\inf_{\mathfrak{m}\in(\mathfrak{f}\star\mathfrak{g})\star\hbar} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \inf_{\mathfrak{n}\in\mathfrak{g}\star\hbar} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{n})\right\}$$

$$\tilde{\varpi}_{\mathfrak{M}}(\mathcal{v}) \leq \max\left\{\sup_{\mathfrak{x}\in(\mathfrak{f}\star\mathfrak{g})\star\hbar} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \sup_{\mathfrak{y}\in\mathfrak{g}\star\hbar} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{y})\right\}$$

($\boldsymbol{IVNFPIhBCKI_2}$) Type 2, if for all $\mathcal{v} \in \mathfrak{f} \star \hbar$,

$$\tilde{\xi}_{\mathfrak{M}}(\mathcal{v}) \geq \min\left\{\sup_{\mathfrak{m}\in(\mathfrak{f}\star\mathfrak{g})\star\hbar} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \inf_{\mathfrak{n}\in\mathfrak{g}\star\hbar} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{n})\right\}$$

$$\tilde{\varpi}_{\mathfrak{M}}(\mathcal{v}) \leq \max\left\{\inf_{\mathfrak{x}\in(\mathfrak{f}\star\mathfrak{g})\star\hbar} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \sup_{\mathfrak{y}\in\mathfrak{g}\star\hbar} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{y})\right\}$$

($\boldsymbol{IVNFPIhBCKI_3}$) Type 3, if for all $\mathcal{v} \in \mathfrak{f} \star \hbar$,

$$\tilde{\xi}_{\mathfrak{M}}(\mathcal{v}) \geq \min\left\{\sup_{\mathfrak{m}\in(\mathfrak{f}\star\mathfrak{g})\star\hbar} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \sup_{\mathfrak{n}\in\mathfrak{g}\star\hbar} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{n})\right\}$$

$$\tilde{\varpi}_{\mathfrak{M}}(\mathcal{v}) \leq \max\left\{\inf_{\mathfrak{x}\in(\mathfrak{f}\star\mathfrak{g})\star\hbar} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \inf_{\mathfrak{y}\in\mathfrak{g}\star\hbar} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{y})\right\}$$

($\boldsymbol{IVNFPIhBCKI_4}$) Type 4, if for all $\mathcal{v} \in \mathfrak{f} \star \hbar$,

$$\tilde{\xi}_{\mathfrak{M}}(\mathcal{v}) \geq \min\left\{\inf_{\mathfrak{m}\in(\mathfrak{f}\star\mathfrak{g})\star\hbar} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \sup_{\mathfrak{n}\in\mathfrak{g}\star\hbar} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{n})\right\}$$

$$\tilde{\varpi}_{\mathfrak{M}}(\mathcal{v}) \leq \max\left\{\sup_{\mathfrak{x}\in(\mathfrak{f}\star\mathfrak{g})\star\hbar} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \inf_{\mathfrak{y}\in\mathfrak{g}\star\hbar} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{y})\right\}$$

for all $\mathfrak{f}, \mathfrak{g}, \hbar \in K$.

Definition 2.8[10]. Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ be an IVIF subset of K. Then $\widetilde{\mathfrak{M}}$ is said to be an IVIFPI-h-BCK-I of

(i) Type 5, if there exists $\mathcal{v} \in \mathfrak{f} \star \hbar$ such that

$$\tilde{\xi}_{\mathfrak{M}}(\mathcal{v}) \geq \min\left\{\inf_{\mathfrak{m}\in(\mathfrak{f}\star\mathfrak{g})\star\hbar} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \inf_{\mathfrak{n}\in\mathfrak{g}\star\hbar} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{n})\right\}$$

$$\tilde{\varpi}_{\mathfrak{M}}(\mathcal{v}) \leq \max\left\{\sup_{\mathfrak{x}\in(\mathfrak{f}\star\mathfrak{g})\star\hbar} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \sup_{\mathfrak{y}\in\mathfrak{g}\star\hbar} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{y})\right\}.$$

(ii) Type 6, if there exists $\mathcal{v} \in \mathfrak{f} \star \hbar$ such that

$$\tilde{\xi}_{\mathfrak{M}}(\mathscr{v}) \geq \min\left\{\sup_{\mathfrak{m}\in(\mathfrak{f}\star\mathfrak{g})\star\hbar} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \sup_{\mathfrak{n}\in\mathfrak{g}\star\hbar} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{n})\right\}$$

$$\tilde{\varpi}_{\mathfrak{M}}(\mathscr{v}) \leq \max\left\{\inf_{\mathfrak{x}\in(\mathfrak{f}\star\mathfrak{g})\star\hbar} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \inf_{\mathfrak{y}\in\mathfrak{g}\star\hbar} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{y})\right\}.$$

(iii) Type 7, if there exists $\mathscr{v} \in \mathfrak{f} \star \hbar$ such that

$$\tilde{\xi}_{\mathfrak{M}}(\mathscr{v}) \geq \min\left\{\inf_{\mathfrak{m}\in(\mathfrak{f}\star\mathfrak{g})\star\hbar} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \sup_{\mathfrak{n}\in\mathfrak{g}\star\hbar} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{n})\right\}$$

$$\tilde{\varpi}_{\mathfrak{M}}(\mathscr{v}) \leq \max\left\{\sup_{\mathfrak{x}\in(\mathfrak{f}\star\mathfrak{g})\star\hbar} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \inf_{\mathfrak{y}\in\mathfrak{g}\star\hbar} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{y})\right\}.$$

(iv) Type 8, if there exists $\mathscr{v} \in \mathfrak{f} \star \hbar$ such that

$$\tilde{\xi}_{\mathfrak{M}}(\mathscr{v}) \geq \min\left\{\sup_{\mathfrak{m}\in(\mathfrak{f}\star\mathfrak{g})\star\hbar} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \inf_{\mathfrak{n}\in\mathfrak{g}\star\hbar} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{n})\right\}$$

$$\tilde{\varpi}_{\mathfrak{M}}(\mathscr{v}) \leq \max\left\{\inf_{\mathfrak{x}\in(\mathfrak{f}\star\mathfrak{g})\star\hbar} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \sup_{\mathfrak{y}\in\mathfrak{g}\star\hbar} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{y})\right\}.$$

for all $\mathfrak{f}, \mathfrak{g}, \hbar \in K$.

III. Interval-Valued Neutrosophic Fuzzy Hyper BCK-Ideals of Hyper BCK-algebras

In the subsequent discussion, the idea of interval-valued fuzzy sets to Neutrosophic fuzzy hyper BCK-ideals in hyper BCK-algebras and related properties are explore; in this article the symbol K will represent a h-BCK-A, unless alternative notation is stated. And also "$\star$" becomes a binary hyper operator Composition.

Definition 3.1 An IVNF $\widetilde{\mathfrak{M}} = \left(\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}}\right)$ in K is said to be an IVNF-h-BCK-I of K if it fulfils

$(\boldsymbol{IVNFhBCKI-1})$ $\mathfrak{f} \ll \mathfrak{g} \Rightarrow \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})$, $\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})$ and $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})$

$(\boldsymbol{IVNFhBCKI-2})$ $\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{m}\in\mathfrak{f}\star\mathfrak{g}} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\right\}$

$(\boldsymbol{IVNFhBCKI-3})$ $\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{k}\in\mathfrak{f}\star\mathfrak{g}} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\right\}$

$(\boldsymbol{IVNFhBCKI-4})$ $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\left\{\sup_{\mathfrak{x}\in\mathfrak{f}\star\mathfrak{g}} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\right\}$

for all $\mathfrak{f}, \mathfrak{g} \in K$.

Example 3.2 Let $K = \{0, m, n\}$ be a set equipped with the binary operation "$\star$" defined by

$\star$	0	m	n
0	$\{0\}$	$\{0\}$	$\{0\}$
m	$\{m\}$	$\{0, m\}$	$\{0, m\}$
n	$\{n\}$	$\{m, n\}$	$\{0, m, n\}$

Then $(K, 0)$ is a h-BCK-A [4]. Define an IVNFS $\widetilde{\mathfrak{M}} = \left(\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}}\right)$ in K by
$\tilde{\xi}_{\mathfrak{M}}(0) = [0.85, 0.9]$, $\tilde{\xi}_{\mathfrak{M}}(m) = [0.55, 0.6]$, $\tilde{\xi}_{\mathfrak{M}}(n) = [0.3, 0.5]$,
$\tilde{\zeta}_{\mathfrak{M}}(0) = [0.75, 0.8]$, $\tilde{\zeta}_{\mathfrak{M}}(m) = [0.45, 0.5]$, $\tilde{\zeta}_{\mathfrak{M}}(n) = [0.2, 0.3]$ and
$\tilde{\varpi}_{\mathfrak{M}}(0) = [0.08, 0.09]$, $\tilde{\varpi}_{\mathfrak{M}}(m) = [0.5, 0.65]$, $\tilde{\varpi}_{\mathfrak{M}}(n) = [0.7, 0.75]$.
Simple verification shows that $\widetilde{\mathfrak{M}} = \left(\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}}\right)$ is an IVNF-h-BCK-I of K.

Definition 3.3 An IVNFS $\widetilde{\mathfrak{M}} = \left(\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}}\right)$ in K is said to be an IVNF-$strong$-h-BCK-I of K if it fulfils

$(\boldsymbol{IVNFShBCKI-1})\ \inf_{\mathfrak{m}\in\mathfrak{f}\star\mathfrak{f}} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\sup_{\mathfrak{n}\in\mathfrak{f}\star\mathfrak{g}} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{n}), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\right\}$,

$(\boldsymbol{IVNFShBCKI-2})\ \inf_{\mathfrak{k}\in\mathfrak{f}\star\mathfrak{f}} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\sup_{\mathfrak{l}\in\mathfrak{f}\star\mathfrak{g}} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{l}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\right\}$ and

$(\boldsymbol{IVNFShBCKI-3})\ \sup_{\mathfrak{x}\in\mathfrak{f}\star\mathfrak{f}} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\left\{\inf_{\mathfrak{y}\in\mathfrak{f}\star\mathfrak{g}} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{y}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\right\}$,

for all $\mathfrak{f}, \mathfrak{g} \in K$.

Example 3.4 Let $K = \{0, m, n\}$ be a set equipped with the binary operation "$\star$" defined by:

$\star$	0	m	n
0	$\{0\}$	$\{0\}$	$\{0\}$
m	$\{m\}$	$\{0\}$	$\{m\}$
n	$\{n\}$	$\{n\}$	$\{0, n\}$

Then $(K, 0)$ is a h-BCK-A [4]. Define an IVNFS $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ in K by
$\tilde{\xi}_{\mathfrak{M}}(0) = [0.75, 0.8]$, $\tilde{\xi}_{\mathfrak{M}}(m) = [0.4, 0.5]$, $\tilde{\xi}_{\mathfrak{M}}(n) = [0.15, 0.2]$,
$\tilde{\zeta}_{\mathfrak{M}}(0) = [0.65, 0.7]$, $\tilde{\zeta}_{\mathfrak{M}}(m) = [0.35, 0.4]$, $\tilde{\zeta}_{\mathfrak{M}}(n) = [0.14, 0.1]$ and
$\tilde{\varpi}_{\mathfrak{M}}(0) = [0.07, 0.08]$, $\tilde{\varpi}_{\mathfrak{M}}(m) = [0.14, 0.18]$, $\tilde{\varpi}_{\mathfrak{M}}(n) = [0.21, 0.26]$.
Simple verification shows that $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ is an IVNF-$strong$-h-BCK-I of K.

Definition 3.5 An IVNF $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ in K is said to be an IVNF-s-$weak$-h-BCK-I of K if it fulfils
$(\boldsymbol{IVNFsWhBCKI-1})$ $\tilde{\xi}_{\mathfrak{M}}(0) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})$, $\tilde{\zeta}_{\mathfrak{M}}(0) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})$ and $\tilde{\varpi}_{\mathfrak{M}}(0) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})$,
$(\boldsymbol{IVNFsWhBCKI-2})$ for every $\mathfrak{f}, \mathfrak{g} \in K$ there exists $\mathfrak{m}, \mathfrak{k}, \mathfrak{x} \in \mathfrak{f} \star \mathfrak{g}$ such that

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\{\tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\},$$

$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\{\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\} \text{ and}$$

$$\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\{\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\}.$$

for all $\mathfrak{f}, \mathfrak{g} \in K$.

Definition 3.6 An IVNF $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ in K is said to be an IVNF-$weak$-h-BCK-I of K if it fulfils

$(\boldsymbol{IVNFWhBCKI-1})$ $\tilde{\xi}_{\mathfrak{M}}(0) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{m}\in\mathfrak{f}\star\mathfrak{g}} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\right\}$,

$(\boldsymbol{IVNFWhBCKI-2})$ $\tilde{\zeta}_{\mathfrak{M}}(0) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{k}\in\mathfrak{f}\star\mathfrak{g}} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\right\}$ and

$(\boldsymbol{IVNFWhBCKI-3})$ $\tilde{\varpi}_{\mathfrak{M}}(0) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\left\{\sup_{\mathfrak{x}\in\mathfrak{f}\star\mathfrak{g}} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\right\}$,

for all $\mathfrak{f}, \mathfrak{g} \in K$.

Theorem 3.7 Every IVNF-s-$weak$-h-BCK-I of K is an IVNF-$weak$-h-BCK-I.
Proof: Let an IVNFS $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ in K be an IVNF-s-$weak$-h-BCK-I of K and let $\mathfrak{f}, \mathfrak{g} \in K$. Then $\exists$ $\mathfrak{m}, \mathfrak{k}, \mathfrak{x} \in \mathfrak{f} \star \mathfrak{g}$ such that
$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\{\tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\}$, $\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\{\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\}$ and $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\{\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\}$
Since,

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \geq \inf_{a\in\mathfrak{f}\star\mathfrak{g}} \tilde{\xi}_{\mathfrak{M}}(a), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{m}) \geq \inf_{b\in\mathfrak{f}\star\mathfrak{g}} \tilde{\zeta}_{\mathfrak{M}}(b) \text{ and } \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{m}) \leq \sup_{c\in\mathfrak{f}\star\mathfrak{g}} \tilde{\varpi}_{\mathfrak{M}}(c),$$

we can conclude that,

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\inf_{a\in\mathfrak{f}\star\mathfrak{g}} \tilde{\xi}_{\mathfrak{M}}(a), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\right\},$$
$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\inf_{b\in\mathfrak{f}\star\mathfrak{g}} \tilde{\zeta}_{\mathfrak{M}}(b), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\right\} \text{ and}$$
$$\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\left\{\sup_{c\in\mathfrak{f}\star\mathfrak{g}} \tilde{\varpi}_{\mathfrak{M}}(c), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\right\}.$$

Definition 3.8 An IVNFS $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ in K is said to fulfil the "inf-sup" property, if for any sub-set $\mathcal{Z}$ of K ∃ $\mathfrak{f}_0, \mathfrak{g}_0, \hbar_0 \in \mathcal{Z}$ such that

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}_0) = \inf_{\mathfrak{f}\in\mathcal{Z}} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}),$$
$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g}_0) = \inf_{\mathfrak{g}\in\mathcal{Z}} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g}) \text{ and}$$
$$\tilde{\varpi}_{\mathfrak{M}}(\hbar_0) = \sup_{\hbar\in\mathcal{Z}} \tilde{\varpi}_{\mathfrak{M}}(\hbar).$$

Constructing a counterexample of an IVNF-s-*weak*-h-BCK-I of K that fails to be an IVNF-*weak*-h-BCK-I of K poses a significant challenge.

Proposition 3.9 Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ be an IVNF-*weak*-h-BCK-I of K. If $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ is fulfils the "inf-sup" property, then $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ is an IVNF-s-*weak*-h-BCK-I of K.

Proof: Suppose that $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ is an IVNF-*weak*-h-BCK-I of K and fulfils the "inf-sup" property. Then there exists $\mathfrak{m}_0, \mathfrak{k}_0, \mathfrak{x}_0 \in \mathfrak{f} \star \mathfrak{g}$ such that

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}_0) \geq \inf_{\mathfrak{m}\in\mathfrak{f}\star\mathfrak{g}} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}_0) \geq \inf_{\mathfrak{k}\in\mathfrak{f}\star\mathfrak{g}} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}) \text{ and } \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}_0) \leq \sup_{\mathfrak{x}\in\mathfrak{f}\star\mathfrak{g}} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}).$$

This leads to the conclusion that

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{m}\in\mathfrak{f}\star\mathfrak{g}} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\right\} = \min\{\tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}_0), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\},$$
$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{k}\in\mathfrak{f}\star\mathfrak{g}} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\right\} = \min\{\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}_0), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\} \text{ and}$$
$$\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\left\{\sup_{\mathfrak{x}\in\mathfrak{f}\star\mathfrak{g}} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\right\} = \max\{\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}_0), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\}.$$

The proof of the theorem is hereby established.

Interestingly, every IVNFS in a finite h-BCK-algebra fulfils the "inf-sup" condition.

This implies that, within the framework of h-BCK-A's, IVNF-*weak*-h-BCK-Is and IVNF-s-*weak*-h-BCK-Is are equivalent notions.

Proposition 3.10 Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ be an IVNF-*strong*-h-BCK-I of K and let $\mathfrak{f}, \mathfrak{g} \in K$.
Then

(i) $\tilde{\xi}_{\mathfrak{M}}(0) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})$, $\tilde{\zeta}_{\mathfrak{M}}(0) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})$ and $\tilde{\varpi}_{\mathfrak{M}}(0) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})$

(ii) $\mathfrak{f} \ll \mathfrak{g} \Rightarrow \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})$ and $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})$

(iii) $\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\{\tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\}$, $\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\{\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\}$ and $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\{\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\}$ for all $\mathfrak{m}, \mathfrak{k}, \mathfrak{x} \in \mathfrak{f} \star \mathfrak{g}$.

Proof: Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ be an IVNF-*strong*-h-BCK-I of K and let $\mathfrak{f}, \mathfrak{g} \in K$.

(i) Since, $0 \in \mathfrak{f} \star \mathfrak{f}$, for all $\mathfrak{f} \in K$, we have

$$\tilde{\xi}_{\mathfrak{M}}(0) \geq \inf_{\mathfrak{m} \in \mathfrak{f} \star \mathfrak{f}} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}),$$
$$\tilde{\zeta}_{\mathfrak{M}}(0) \geq \inf_{\mathfrak{t} \in \mathfrak{f} \star \mathfrak{g}} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{t}) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \text{ and}$$
$$\tilde{\varpi}_{\mathfrak{M}}(0) \leq \sup_{\mathfrak{x} \in \mathfrak{f} \star \mathfrak{g}} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}),$$

for all $\mathfrak{f} \in K$.
Hence, $\tilde{\xi}_{\mathfrak{M}}(0) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})$, $\tilde{\zeta}_{\mathfrak{M}}(0) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})$ and $\tilde{\varpi}_{\mathfrak{M}}(0) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})$, for all $\mathfrak{f}, \mathfrak{g} \in K$.
(ii) Let $\mathfrak{f}, \mathfrak{g} \in K$ be such that $\mathfrak{f} \ll \mathfrak{g}$ then $0 \in \mathfrak{f} \star \mathfrak{f}$ and so

$$\tilde{\xi}_{\mathfrak{M}}(0) \leq \sup_{a \in \mathfrak{f} \star \mathfrak{g}} \tilde{\xi}_{\mathfrak{M}}(a), \tilde{\zeta}_{\mathfrak{M}}(0) \leq \sup_{b \in \mathfrak{f} \star \mathfrak{g}} \tilde{\zeta}_{\mathfrak{M}}(b) \text{ and } \tilde{\varpi}_{\mathfrak{M}}(0) \geq \inf_{c \in \mathfrak{f} \star \mathfrak{g}} \tilde{\varpi}_{\mathfrak{M}}(c)$$

By virtue of (i), we have that

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\sup_{a \in \mathfrak{f} \star \mathfrak{g}} \tilde{\xi}_{\mathfrak{M}}(a), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\right\} \geq \min\{\tilde{\xi}_{\mathfrak{M}}(0), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\} = \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g}),$$
$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\sup_{b \in \mathfrak{f} \star \mathfrak{g}} \tilde{\zeta}_{\mathfrak{M}}(b), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\right\} \geq \min\{\tilde{\zeta}_{\mathfrak{M}}(0), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\} = \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g}) \text{ and}$$
$$\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\left\{\inf_{c \in \mathfrak{f} \star \mathfrak{g}} \tilde{\varpi}_{\mathfrak{M}}(c), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\right\} \leq \max\{\tilde{\varpi}_{\mathfrak{M}}(0), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\} = \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g}).$$

Hence, if $\mathfrak{f} \ll \mathfrak{g} \Rightarrow \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})$ and $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})$.
(iii) Let $\mathfrak{f}, \mathfrak{g} \in K$, since

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\sup_{a \in \mathfrak{f} \star \mathfrak{g}} \tilde{\xi}_{\mathfrak{M}}(a), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\right\} \geq \min\{\tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\},$$
$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\sup_{b \in \mathfrak{f} \star \mathfrak{g}} \tilde{\zeta}_{\mathfrak{M}}(b), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\right\} \geq \min\{\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{t}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\} \text{ and}$$
$$\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\left\{\sup_{c \in \mathfrak{f} \star \mathfrak{g}} \tilde{\varpi}_{\mathfrak{M}}(c), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\right\} \leq \max\{\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\},$$

for all $\mathfrak{m}, \mathfrak{t}, \mathfrak{x} \in \mathfrak{f} \star \mathfrak{g}$.

Theorem 3.11 Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ is an IVNF-$strong$-h-BCK-I of K and let $\mathfrak{f}, \mathfrak{g} \in K$. Then

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{m} \in \mathfrak{f} \star \mathfrak{g}} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\right\}, \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{t} \in \mathfrak{f} \star \mathfrak{g}} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{t}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\right\} \text{ and}$$
$$\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\left\{\sup_{\mathfrak{x} \in \mathfrak{f} \star \mathfrak{g}} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\right\}, \forall\, \mathfrak{f}, \mathfrak{g} \in K.$$

Proof: For any $\mathfrak{f}, \mathfrak{g} \in K$, we have

$$\sup_{\mathfrak{m} \in \mathfrak{f} \star \mathfrak{g}} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \geq \sup_{\mathfrak{m} \in \mathfrak{f} \star \mathfrak{g}} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}),$$
$$\sup_{\mathfrak{t} \in \mathfrak{f} \star \mathfrak{g}} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{t}) \geq \sup_{\mathfrak{t} \in \mathfrak{f} \star \mathfrak{g}} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{t}) \text{ and}$$
$$\inf_{\mathfrak{x} \in \mathfrak{f} \star \mathfrak{g}} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}) \leq \inf_{\mathfrak{x} \in \mathfrak{f} \star \mathfrak{g}} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}),$$

for all $\mathfrak{f} \in K$.
This yields the definition that we obtain as

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\sup_{\mathfrak{m} \in \mathfrak{f} \star \mathfrak{g}} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\right\} \geq \min\left\{\inf_{\mathfrak{m} \in \mathfrak{f} \star \mathfrak{g}} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\right\},$$
$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\sup_{\mathfrak{t} \in \mathfrak{f} \star \mathfrak{g}} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{t}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\right\} \geq \min\left\{\inf_{\mathfrak{t} \in \mathfrak{f} \star \mathfrak{g}} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{t}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\right\} \text{ and}$$
$$\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\left\{\inf_{\mathfrak{x} \in \mathfrak{f} \star \mathfrak{g}} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\right\} \leq \max\left\{\sup_{\mathfrak{x} \in \mathfrak{f} \star \mathfrak{g}} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\right\},$$

for all $\mathfrak{f}, \mathfrak{g} \in K$.

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Corollary 3.12 Every IVNF-*strong*-h-BCK-I is both an IVNF-s-*weak*-h-BCK-I (and hence an IVNF-*weak*-h-BCK-I) and an IVNF-h-BCK-I.

IV. Interval-Valued Neutrosophic Fuzzy Implicative Hyper BCK-Ideals of Hyper BCK-algebras

In the subsequent discussion, the idea of interval-valued fuzzy sets to Neutrosophic fuzzy implicative hyper BCK-ideals in hyper BCK-algebras and related properties are explained.

Definition 4.1 Let $\mathfrak{M} = \left(\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}}\right)$ be an IVNFS on K and $\tilde{\xi}_{\mathfrak{M}}(0) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f})$, $\tilde{\zeta}_{\mathfrak{M}}(0) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f})$, $\tilde{\varpi}_{\mathfrak{M}}(0) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f})$, ∀ $\mathfrak{f}, \mathfrak{g} \in K$. Then $\mathfrak{M}$ is said to be an

(i) IVNF-*weak*-I-h-BCK-I of K, if

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{m}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\hbar)\right\},$$

$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{t}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{t}), \tilde{\zeta}_{\mathfrak{M}}(\hbar)\right\} \text{ and}$$

$$\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\left\{\sup_{\mathfrak{x}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\hbar)\right\}$$

(ii) IVNF-I-h-BCK-I of K, if

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\sup_{\mathfrak{m}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\hbar)\right\},$$

$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\sup_{\mathfrak{t}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{t}), \tilde{\zeta}_{\mathfrak{M}}(\hbar)\right\} \text{ and}$$

$$\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\left\{\inf_{\mathfrak{x}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\hbar)\right\}$$

∀ $\mathfrak{f}, \mathfrak{g}, \hbar \in K$.

Theorem 4.2 Every interval-valued Neutrosophic fuzzy implicative hyper BCK-ideal (NFI-h-BCK-I) of K is an interval-valued Neutrosophic fuzzy *weak* implicative hyper BCK-ideal(NF-*weak*-I-h-BCK-I).

Proof: It is assumed that $\mathfrak{M} = (\xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ is an IVNFI-h-BCK-I of K and for $\mathfrak{f}, \mathfrak{g}, \hbar \in K$. Since

$$\inf_{\mathfrak{m}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \leq \sup_{\mathfrak{m}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}),$$

$$\inf_{\mathfrak{t}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{t}) \leq \sup_{\mathfrak{t}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{t}) \text{ and}$$

$$\sup_{\mathfrak{x}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}) \geq \inf_{\mathfrak{x}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}).$$

Accordingly,

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\sup_{\mathfrak{m}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\hbar)\right\} \geq \min\left\{\inf_{\mathfrak{m}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\hbar)\right\},$$

$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\sup_{\mathfrak{t}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{t}), \tilde{\zeta}_{\mathfrak{M}}(\hbar)\right\} \geq \min\left\{\inf_{\mathfrak{t}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{t}), \tilde{\zeta}_{\mathfrak{M}}(\hbar)\right\} \text{ and}$$

$$\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\left\{\inf_{\mathfrak{x}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \varpi_{\mathfrak{M}}(\hbar)\right\} \leq \max\left\{\sup_{\mathfrak{x}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \varpi_{\mathfrak{M}}(\hbar)\right\}.$$

Hence, $\widetilde{\mathfrak{M}} = \left(\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}}\right)$ is an IVNF-*weak*-I-h-BCK-I of K.

Example 4.3 The set $K = \{0, m, n\}$ is given and its operation is specified in the following table.

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$\star$	0	m	n
0	$\{0\}$	$\{0\}$	$\{0\}$
m	$\{m\}$	$\{0,m\}$	$\{0,m\}$
n	$\{n\}$	$\{m\}$	$\{0,m\}$

Then $(K,\ \star)$ is a h-BCK-A [3]. Define an IVNFS $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ on K by
$\tilde{\xi}_{\mathfrak{M}}(0) = \widetilde{0.9}$, $\tilde{\xi}_{\mathfrak{M}}(m) = \widetilde{0.5}$, $\tilde{\xi}_{\mathfrak{M}}(n) = \widetilde{0.7}$, $\tilde{\zeta}_{\mathfrak{M}}(0) = \widetilde{0.8}$, $\tilde{\zeta}_{\mathfrak{M}}(m) = \widetilde{0.4}$, $\tilde{\zeta}_{\mathfrak{M}}(n) = \widetilde{0.6}$ and
$\tilde{\varpi}_{\mathfrak{M}}(0) = \widetilde{0.5}$, $\tilde{\varpi}_{\mathfrak{M}}(m) = \widetilde{0.9}$, $\tilde{\varpi}_{\mathfrak{M}}(n) = \widetilde{0.3}$.
The set $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ forms an IVNF-*weak*-I-h-BCK-I of $\mathfrak{H}$. However, it fails to be an IVNF-I-h-BCK-I of K, due to

$$\tilde{\xi}_{\mathfrak{M}}(m) = \widetilde{0.5} < \widetilde{0.9} = \tilde{\xi}_{\mathfrak{M}}(0) = \min\left\{\sup_{\mathcal{v}\in(m\star 0)\star(m\star m)} \tilde{\xi}_{\mathfrak{M}}(\mathcal{v}), \tilde{\xi}_{\mathfrak{M}}(0)\right\},$$

$$\tilde{\zeta}_{\mathfrak{M}}(m) = \widetilde{0.4} < \widetilde{0.8} = \tilde{\zeta}_{\mathfrak{M}}(0) = \min\left\{\sup_{\mathcal{v}\in(m\star 0)\star(m\star m)} \tilde{\zeta}_{\mathfrak{M}}(\mathcal{v}), \tilde{\zeta}_{\mathfrak{M}}(0)\right\} \text{ and}$$

$$\tilde{\varpi}_{\mathfrak{M}}(m) = \widetilde{0.9} > \widetilde{0.5} = \tilde{\varpi}_{\mathfrak{M}}(0) = \max\left\{\inf_{\mathcal{v}\in(m\star 0)\star(m\star m)} \tilde{\varpi}_{\mathfrak{M}}(\mathcal{v}), \tilde{\varpi}_{\mathfrak{M}}(0)\right\}.$$

Theorem 4.4

1. Every IVNFI-h-BCK-I of K is an IVNF-*strong*-h-BCK-I.
2. Every IVNF-*weak*-I-h-BCK-I of K is an IVNF-*weak*-h-BCK-I.

Proof: 1. Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ be an IVNFI-h-BCK-I of K.
Inserting, $\mathfrak{g} = 0$ and $\hbar = \mathfrak{g}$ in Definition 4.1 (ii), we obtain

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\sup_{\mathfrak{m}\in(\mathfrak{f}\star\mathfrak{g})\star(0\star\mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\right\} = \min\left\{\sup_{\mathfrak{m}\in(\mathfrak{f}\star\mathfrak{g})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\right\},$$

$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\sup_{\mathfrak{k}\in(\mathfrak{f}\star\mathfrak{g})\star(0\star\mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\right\} = \min\left\{\sup_{\mathfrak{k}\in(\mathfrak{f}\star\mathfrak{g})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\right\} \text{ and}$$

$$\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\left\{\inf_{\mathfrak{x}\in(\mathfrak{f}\star\mathfrak{g})\star(0\star\mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\right\} = \max\left\{\inf_{\mathfrak{x}\in(\mathfrak{f}\star\mathfrak{g})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\right\}, \ldots.(i).$$

Initially, we prove that, for $\mathfrak{f}, \mathfrak{g} \in K$, if $\mathfrak{f} \ll \mathfrak{g} \Rightarrow \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})$, $\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})$ and $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})$
For this, let $\mathfrak{f}, \mathfrak{g} \in K$ be such that $\mathfrak{f} \ll \mathfrak{g}$, then $0 \in \mathfrak{f} \star \mathfrak{g}$ and thus, by (i), we obtain

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\sup_{\mathfrak{m}\in(\mathfrak{f}\star\mathfrak{g})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\right\} = \min\{\tilde{\xi}_{\mathfrak{M}}(0), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\} = \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g}),$$

$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\sup_{\mathfrak{k}\in(\mathfrak{f}\star\mathfrak{g})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\right\} = \min\{\tilde{\zeta}_{\mathfrak{M}}(0), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\} = \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g}) \text{ and}$$

$$\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\left\{\inf_{\mathfrak{x}\in(\mathfrak{f}\star\mathfrak{g})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\right\} = \max\{\tilde{\varpi}_{\mathfrak{M}}(0), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\} = \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g}), \ldots.(ii).$$

Let $\mathfrak{f} \in K$ and $\mathfrak{m} \in \mathfrak{f} \star \mathfrak{f}$. Since, $\mathfrak{f} \star \mathfrak{f} \ll \mathfrak{f}$ then $\mathfrak{m} \ll \mathfrak{f}$ for all $\mathfrak{m} \in \mathfrak{f} \star \mathfrak{f}$ and thus, by (ii), we have $\tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f})$, $\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{m}) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f})$ and $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{m}) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f})$, $\forall\, \mathfrak{m} \in \mathfrak{f} \star \mathfrak{f}$. Therefore,

$$\inf_{\mathfrak{m}\in\mathfrak{f}\star\mathfrak{f}} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}),\ \inf_{\mathfrak{m}\in\mathfrak{f}\star\mathfrak{f}} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{m}) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \text{ and } \sup_{\mathfrak{m}\in\mathfrak{f}\star\mathfrak{f}} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{m}) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}), \ldots.(iii).$$

Taking (i) and (iii) into account, we obtain

$$\inf_{\mathfrak{m}\in\mathfrak{f}\star\mathfrak{f}} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\sup_{\mathfrak{n}\in(\mathfrak{f}\star\mathfrak{g})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{n}), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\right\},$$

$$\inf_{\mathfrak{m}\in\mathfrak{f}\star\mathfrak{f}} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{m}) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\sup_{\mathfrak{l}\in(\mathfrak{f}\star\mathfrak{g})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{l}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\right\} \text{ and}$$

$$\sup_{\mathfrak{m}\in\mathfrak{f}\star\mathfrak{f}} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{m}) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\left\{\inf_{\mathfrak{y}\in(\mathfrak{f}\star\mathfrak{g})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{y}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\right\},$$

for all $\mathfrak{f}, \mathfrak{g} \in K$.

Hence $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ satisfies the conditions of an IVNF-$strong$-h-BCK-I of K.

2. Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ is an IVNF-$weak$-I-h-BCK-I of K.
Inserting, $\mathfrak{g} = 0$ and $\hbar = \mathfrak{g}$ in Definition 4.1 (i), we obtain

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{m}\in(\mathfrak{f}\star\mathfrak{g})\star(0\star\mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\right\} = \min\left\{\inf_{\mathfrak{m}\in(\mathfrak{f}\star\mathfrak{g})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\right\},$$

$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{t}\in(\mathfrak{f}\star\mathfrak{g})\star(0\star\mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{t}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\right\} = \min\left\{\inf_{\mathfrak{t}\in(\mathfrak{f}\star\mathfrak{g})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{t}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\right\} \text{ and}$$

$$\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\left\{\sup_{\mathfrak{x}\in(\mathfrak{f}\star\mathfrak{g})\star(0\star\mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\right\} = \max\left\{\sup_{\mathfrak{x}\in(\mathfrak{f}\star\mathfrak{g})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\right\}.$$

Hence,

$$\tilde{\xi}_{\mathfrak{M}}(0) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{m}\in(\mathfrak{f}\star\mathfrak{g})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\right\},$$

$$\tilde{\zeta}_{\mathfrak{M}}(0) \geq \zeta_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{t}\in(\mathfrak{f}\star\mathfrak{g})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{t}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\right\} \text{ and}$$

$$\tilde{\varpi}_{\mathfrak{M}}(0) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\left\{\sup_{\mathfrak{x}\in(\mathfrak{f}\star\mathfrak{g})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\right\},$$

for all $\mathfrak{f}, \mathfrak{g} \in K$.
Thus $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ is an IVNF-$weak$-I-h-BCK-I of K.

Example 4.5 Let $K = \{0, m, n, \mathfrak{k}\}$ be the set. The Cayley table for K is given by:

$\star$	0	m	n	$\mathfrak{k}$
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
m	$\{m\}$	$\{0\}$	$\{0\}$	$\{0\}$
n	$\{n\}$	$\{n\}$	$\{0\}$	$\{0\}$
$\mathfrak{k}$	$\{\mathfrak{k}\}$	$\{\mathfrak{k}\}$	$\{n, \mathfrak{k}\}$	$\{0, n, \mathfrak{k}\}$

Then $(K, \star)$ is a h-BCK-A [3]. An IVNFS $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ in K is defined as
$\tilde{\xi}_{\mathfrak{M}}(0) = \tilde{\xi}_{\mathfrak{M}}(m) = \widetilde{0.8}$, $\tilde{\xi}_{\mathfrak{M}}(n) = \tilde{\xi}_{\mathfrak{M}}(\mathfrak{k}) = \widetilde{0.3}$, $\tilde{\zeta}_{\mathfrak{M}}(0) = \tilde{\zeta}_{\mathfrak{M}}(m) = \widetilde{0.6}$, $\tilde{\zeta}_{\mathfrak{M}}(n) = \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}) = \widetilde{0.2}$, and $\tilde{\varpi}_{\mathfrak{M}}(0) = \tilde{\varpi}_{\mathfrak{M}}(m) = \widetilde{0.1}$, $\tilde{\varpi}_{\mathfrak{M}}(n) = \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{k}) = \widetilde{0.4}$.
As a result $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ forms an IVNF-$strong$-h-BCK-I (and therefore an IVNF-$weak$-h-BCK-I) However, it does not qualify as an IVNFI (or an IVNF-$weak$-I) hyper BCK-I of K, because

$$\tilde{\xi}_{\mathfrak{M}}(n) = \widetilde{0.3} < \widetilde{0.8} = \tilde{\xi}_{\mathfrak{M}}(0) = \min\left\{\inf_{\mathcal{v}\in(n\star 0)\star(\mathfrak{k}\star n)} \tilde{\xi}_{\mathfrak{M}}(\mathcal{v}), \tilde{\xi}_{\mathfrak{M}}(0)\right\},$$

$$\tilde{\zeta}_{\mathfrak{M}}(n) = \widetilde{0.2} < \widetilde{0.6} = \tilde{\zeta}_{\mathfrak{M}}(0) = \min\left\{\inf_{\mathcal{v}\in(n\star 0)\star(\mathfrak{k}\star n)} \tilde{\zeta}_{\mathfrak{M}}(\mathcal{v}), \tilde{\zeta}_{\mathfrak{M}}(0)\right\} \text{ and}$$

$$\tilde{\varpi}_{\mathfrak{M}}(n) = \widetilde{0.4} > \widetilde{0.1} = \tilde{\varpi}_{\mathfrak{M}}(0) = \max\left\{\sup_{\mathcal{v}\in(n\star 0)\star(\mathfrak{k}\star n)} \tilde{\varpi}_{\mathfrak{M}}(\mathcal{v}), \tilde{\varpi}_{\mathfrak{M}}(0)\right\}.$$

Therefore, Theorem 4.4 is not reversible in general.

Theorem 4.6 Assuming $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ is an IVNF subset of K, we obtain the following
(1) $\widetilde{\mathfrak{M}}$ is an IVNF-$weak$-I-h-BCK-I of $K \Leftrightarrow \forall\, \tilde{s}, \tilde{t}, \tilde{v} \in \mathbb{A}[0,1]$, $\mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s}) \neq \emptyset$, $\mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t}) \neq \emptyset$ and $\mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v}) \neq \emptyset$ are $weak$-I-h-BCK-Is of K.
(2) If $\widetilde{\mathfrak{M}}$ is an IVNF-I-h-BCK-I of K, then $\forall\, \tilde{s}, \tilde{t}, \tilde{v} \in \mathbb{A}[0,1]$, $\mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s}) \neq \emptyset$, $\mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t}) \neq \emptyset$ and $\mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v}) \neq \emptyset$ are I-h-BCK-Is of K.
(3) If $\forall\, \tilde{s}, \tilde{t}, \tilde{v} \in \mathbb{A}[0,1]$, $\mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s}) \neq \emptyset$, $\mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t}) \neq \emptyset$ and $\mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v}) \neq \emptyset$ are S-$reflexive$-I-h-BCK-Is of K, and $\widetilde{\mathfrak{M}}$ satisfies the "sup-inf" property, then $\widetilde{\mathfrak{M}}$ is an IVNF-I-h-BCK-I of K.

Proof: (1) Suppose that $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ is an IVNF-*weak*-I-h-BCK-I of K.
Let $\tilde{s}, \tilde{t}, \tilde{v} \in \mathbb{A}[0,1]$ and $\mathfrak{f}, \mathfrak{g}, \hbar \in K$ be such that $(\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f}) \subseteq \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$ and $\hbar \in \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$.
Then $\mathfrak{m} \in \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$ for all $\mathfrak{m} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f}) \Rightarrow \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \geq \tilde{s}$, for all $\mathfrak{m} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})$ and $\tilde{\xi}_{\mathfrak{M}}(\hbar) \geq \tilde{s}$,

$$\inf_{\mathfrak{m} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \geq \tilde{s} \text{ and } \tilde{\xi}_{\mathfrak{M}}(\hbar) \geq \tilde{s}.$$

As assumed,

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{m} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \Rightarrow \tilde{\xi}_{\mathfrak{M}}(\hbar)\right\} \geq \min\{\tilde{s}, \tilde{s}\} = \tilde{s}.$$

$\Rightarrow \mathfrak{f} \in \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$.

Let $\mathfrak{f}, \mathfrak{g}, \hbar \in K$ be such that $(\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f}) \subseteq \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$ and $\hbar \in \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$.
Then $\mathfrak{k} \in \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$ for all $\mathfrak{m} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f}) \Rightarrow \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}) \geq \tilde{t}$, for all $\mathfrak{k} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})$ and $\tilde{\zeta}_{\mathfrak{M}}(\hbar) \geq \tilde{t}$,

$$\inf_{\mathfrak{k} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}) \geq \tilde{t} \text{ and } \tilde{\zeta}_{\mathfrak{M}}(\hbar) \geq \tilde{t}.$$

As assumed,

$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\inf_{\mathfrak{k} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}) \Rightarrow \tilde{\zeta}_{\mathfrak{M}}(\hbar)\right\} \geq \min\{\tilde{t}, \tilde{t}\} = \tilde{t}.$$

$\Rightarrow \mathfrak{f} \in \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$.

Let $\mathfrak{f}, \mathfrak{g}, \hbar \in K$ be such that $(\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f}) \subseteq \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$ and $\hbar \in \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$.
Then $\mathfrak{x} \in \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$ for all $\mathfrak{m} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f}) \Rightarrow \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}) \leq \tilde{v}$, for all $\mathfrak{x} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})$ and $\tilde{\varpi}_{\mathfrak{M}}(\hbar) \leq \tilde{v}$,

$$\sup_{\mathfrak{x} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}) \leq \tilde{v} \text{ and } \tilde{\varpi}_{\mathfrak{M}}(\hbar) \leq \tilde{v}.$$

As assumed,

$$\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\left\{\sup_{\mathfrak{x} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}) \Rightarrow \tilde{\varpi}_{\mathfrak{M}}(\hbar)\right\} \leq \max\{\tilde{v}, \tilde{v}\} = \tilde{v}.$$

$\Rightarrow \mathfrak{f} \in \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$.

Hence, $\mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$, $\mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$ and $\mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$ are *weak*-I-h-BCK-Is of K, $\forall\ \tilde{s}, \tilde{t}, \tilde{v} \in \mathbb{A}[0,1]$.
In the converse direction, assume that, let $\forall\ \tilde{s}, \tilde{t}, \tilde{v} \in \mathbb{A}[0,1]$, $\mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s}) \neq \emptyset$, $\mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t}) \neq \emptyset$ and $\mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v}) \neq \emptyset$ are *weak*-I-h-BCK-Is of K and $\mathfrak{f}, \mathfrak{g}, \hbar \in K$ and put

$$\tilde{s} = \min\left\{\inf_{\mathfrak{m} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\hbar)\right\}.$$

Then

$$\inf_{\mathfrak{m} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \geq \tilde{s} \text{ and } \tilde{\xi}_{\mathfrak{M}}(\hbar) \geq \tilde{s}.$$

So we have $\tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \geq \tilde{s}$ for all $\mathfrak{m} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})$ and $\xi_{\mathfrak{M}}(\hbar) \geq \tilde{s}$.
Therefore, $\mathfrak{m} \in \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$ for all $\mathfrak{m} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})$ and $\hbar \in \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$.
i.e., $(\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f}) \subseteq \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$, $\hbar \in \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$ and so by hypothesis,

$$\mathfrak{f} \in \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s}) = \min\left\{\inf_{\mathfrak{m} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\hbar)\right\}.$$

Thus

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \geq \tilde{s} = \min\left\{\inf_{\mathfrak{m} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\hbar)\right\}.$$

Let $\mathfrak{f}, \mathfrak{g}, \hbar \in K$ and put

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$$\tilde{t} = \min\left\{\inf_{\mathfrak{k}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}), \tilde{\zeta}_{\mathfrak{M}}(\hbar)\right\}.$$

Then

$$\inf_{\mathfrak{k}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}) \geq \tilde{t} \text{ and } \tilde{\zeta}_{\mathfrak{M}}(\hbar) \geq \tilde{t}.$$

So we have $\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}) \geq \tilde{t}$ for all $\mathfrak{k} \in (\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})$ and $\tilde{\zeta}_{\mathfrak{M}}(\hbar) \geq \tilde{t}$. Therefore, $\mathfrak{k} \in \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\, \tilde{t})$ for all $\mathfrak{k} \in (\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})$ and $\hbar \in \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\, \tilde{t})$. **i.e.,** $(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f}) \subseteq \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\, \tilde{t})$, $\hbar \in \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\, \tilde{t})$ and so by hypothesis,

$$\mathfrak{f} \in \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\, \tilde{t}) = \min\left\{\inf_{\mathfrak{k}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}), \tilde{\zeta}_{\mathfrak{M}}(\hbar)\right\}.$$

Thus

$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}) \geq \tilde{t} = \min\left\{\inf_{\mathfrak{k}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}), \tilde{\zeta}_{\mathfrak{M}}(\hbar)\right\}.$$

Let $\mathfrak{f}, \mathfrak{g}, \hbar \in K$ and put

$$\tilde{v} = \max\left\{\sup_{\mathfrak{x}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\hbar)\right\}.$$

Then

$$\sup_{\mathfrak{x}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}) \leq \tilde{v} \text{ and } \tilde{\varpi}_{\mathfrak{M}}(\hbar) \leq \tilde{v}.$$

So we have $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}) \leq \tilde{v}$ for all $\mathfrak{x} \in (\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})$ and $\tilde{\varpi}_{\mathfrak{M}}(\hbar) \leq \tilde{v}$.
Therefore, $\mathfrak{x} \in \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\, \tilde{v})$ for all $\mathfrak{x} \in (\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})$ and $\hbar \in \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\, \tilde{v})$.
i.e., $(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f}) \subseteq \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\, \tilde{v})$, $\hbar \in \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\, \tilde{v})$ and so by hypothesis,

$$\mathfrak{f} \in \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\, \tilde{v}) = \max\left\{\sup_{\mathfrak{x}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\hbar)\right\}.$$

Thus

$$\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}) \leq \tilde{v} = \max\left\{\sup_{\mathfrak{x}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(\hbar)\right\}.$$

Hence, $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ is an IVNF-$weak$-I-h-BCK-I of K.

(2) Given that $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ is an IVNF-I-h-BCK-I of K.
By Theorem 4.4(1), $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ is an IVNF-$strong$-h-BCK-I of K and so it is an IVNF-h-BCK-I of $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ is an IVNF-h-BCK-I of K by Theorem 3.17[4], for all $\tilde{s}, \tilde{t}, \tilde{v} \in \mathbb{A}[0,1]$,
$\mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\, \tilde{s}) \neq \emptyset$, $\mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\, \tilde{t}) \neq \emptyset$ and $\mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\, \tilde{v}) \neq \emptyset$ are h-BCK-Is of K. By Theorem 4.6(ii) [3], it is enough to show that, let $\mathfrak{f}, \mathfrak{g}, \hbar \in K$ and if $\mathfrak{f}\star(\mathfrak{g}\star\mathfrak{f}) \ll \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\, \tilde{s})$, $\mathfrak{f}\star(\mathfrak{g}\star\mathfrak{f}) \ll \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\, \tilde{t})$ and $\mathfrak{f}\star(\mathfrak{g}\star\mathfrak{f}) \ll \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\, \tilde{v})$, then
$\mathfrak{f} \in \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\, \tilde{s}) \cap \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\, \tilde{t}) \cap \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\, \tilde{v})$. For this, let $\mathfrak{f}\star(\mathfrak{g}\star\mathfrak{f}) \ll \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\, \tilde{s})$ for $\mathfrak{f}, \mathfrak{g} \in K$. Then for all
$\mathfrak{m} \in \mathfrak{f}\star(\mathfrak{g}\star\mathfrak{f})$ there exists $\mathfrak{n} \in \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\, \tilde{s})$ such that $\mathfrak{m} \ll \mathfrak{n}$, we have $\tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{n}) \geq \tilde{s} \Rightarrow$
$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \geq \tilde{s}$ for all $\mathfrak{m} \in \mathfrak{f}\star(\mathfrak{g}\star\mathfrak{f}) \Rightarrow$

$$\sup_{\mathfrak{m}\in\mathfrak{f}\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \geq \tilde{s}.$$

Hence, as assumed,

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\sup_{\mathfrak{m}\in(\mathfrak{f}\star 0)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(0)\right\} = \sup_{\mathfrak{m}\in\mathfrak{f}\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \geq \tilde{s}$$

i.e., $\mathfrak{f} \in \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\, \tilde{s})$.

Let $\mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f}) \ll \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$ for $\mathfrak{f}, \mathfrak{g} \in K$. Then for all $\mathfrak{k} \in \mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f})$ there exists $\mathfrak{l} \in \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$ such that $\mathfrak{k} \ll \mathfrak{l}$, we have $\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{l}) \geq \tilde{t} \Rightarrow \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}) \geq \tilde{t}$ for all $\mathfrak{k} \in \mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f}) \Rightarrow$

$$\sup_{\mathfrak{k} \in \mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}) \geq \tilde{t}.$$

Hence, as assumed,

$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\left\{\sup_{\mathfrak{k} \in (\mathfrak{f} \star 0) \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}), \tilde{\zeta}_{\mathfrak{M}}(0)\right\} = \sup_{\mathfrak{k} \in \mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}) \geq \tilde{t}$$

i.e., $\mathfrak{f} \in \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$.

Let $\mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f}) \ll \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$ for $\mathfrak{f}, \mathfrak{g} \in K$. Then for all $\mathfrak{x} \in \mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f})$ there exists $\mathfrak{y} \in \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$ such that $\mathfrak{x} \ll \mathfrak{y}$, we have $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{y}) \leq \tilde{v} \Rightarrow \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}) \leq \tilde{v}$ for all $\mathfrak{x} \in \mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f}) \Rightarrow$

$$\inf_{\mathfrak{x} \in \mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}) \leq \tilde{v}.$$

Hence, as assumed,

$$\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\left\{\inf_{\mathfrak{x} \in (\mathfrak{f} \star 0) \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}), \tilde{\varpi}_{\mathfrak{M}}(0)\right\} = \inf_{\mathfrak{x} \in \mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}) \leq \tilde{v}$$

i.e., $\mathfrak{f} \in \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$.
Therefore, $\mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$, $\mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$ and $\mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$ qualify as I-h-BCK-Is of K, holding $\forall\ \tilde{s}, \tilde{t}, \tilde{v} \in \mathbb{A}[0,1]$.

(3) Assume that, $\forall\ \tilde{s}, \tilde{t}, \tilde{v} \in \mathbb{A}[0,1]$, $\mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$, $\mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$ and $\mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$ possess the property of S-$reflexive$-I-h-BCK-Is of K.
Let $\mathfrak{f}, \mathfrak{g}, \hbar \in K$. Set

$$\tilde{s} = \min\left\{\sup_{\mathfrak{m} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\hbar)\right\} \Rightarrow \sup_{\mathfrak{m} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \geq \tilde{s} \text{ and } \tilde{\xi}_{\mathfrak{M}}(\hbar) \geq \tilde{s},$$

The "sup" property of $\tilde{\xi}_{\mathfrak{M}}$ implies the existence of $\mathfrak{m}_0 \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})$ such that

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}_0) = \sup_{\mathfrak{m} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \geq \tilde{s}$$

Therefore, $\mathfrak{m}_0 \in \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$, which by $(hBCK2)$, implies $\big((\mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f})) \star \hbar\big) \cap \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s}) = ((\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})) \cap \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s}) \neq \emptyset$, then there exists $\mathfrak{m} \in (\mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f}))$ such that $(\mathfrak{m} \star \hbar) \cap \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s}) \neq \emptyset$. According to Theorem 4.6(i)[3], $\mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$ qualifies as a h-BCK-I of K.
The S-reflexivity of $\mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$ allows us to invoke Theorem 2.3(i)[3] , which establishes that $\mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$ is a reflexive-h-BCK-I of K. Hence, $\mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$ is a $strong$-h-BCK-I of K.
Since, $\tilde{\xi}_{\mathfrak{M}}(\hbar) \geq \tilde{s}$ implies $\hbar \in \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$, $(\mathfrak{m} \star \hbar) \cap \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s}) \neq \emptyset$ and $\hbar \in \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$, then $\mathfrak{f} \in \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$ and so $(\mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f})) \cap \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s}) \neq \emptyset$. The reflexivity of $\mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$ and Theorem 3.5(ii)[3], together imply $\mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f}) \ll \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$. Since $\mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$ is an I-h-BCK-I of K we deduce $\mathfrak{f} \in \mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$, and hence,

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}) \geq \tilde{s} = \min\left\{\sup_{\mathfrak{m} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\xi}_{\mathfrak{M}}(\mathfrak{m}), \tilde{\xi}_{\mathfrak{M}}(\hbar)\right\}.$$

With, $\mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$ being an S-$reflexive$-I-h-BCK-Is of K. Let $\mathfrak{f}, \mathfrak{g}, \hbar \in K$. Put

$$\tilde{t} = \min\left\{\sup_{\mathfrak{k} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}), \tilde{\zeta}_{\mathfrak{M}}(\hbar)\right\} \Rightarrow \sup_{\mathfrak{k} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}) \geq \tilde{t} \text{ and } \tilde{\zeta}_{\mathfrak{M}}(\hbar) \geq \tilde{t},$$

The "sup" property of $\tilde{\zeta}_{\mathfrak{M}}$ implies the existence of $\mathfrak{k}_0 \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})$ such that

$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}_0) = \sup_{\mathfrak{k} \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}) \geq \tilde{t}$$

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Therefore $\mathfrak{k}_0 \in \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$, which by $(hBCK2)$, implies
$\left(\left(\mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f})\right) \star \hbar\right) \cap \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t}) = \left((\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})\right) \cap \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t}) \neq \emptyset$, then there exists $\mathfrak{k} \in \left(\mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f})\right)$ such that $(\mathfrak{k} \star \hbar) \cap \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t}) \neq \emptyset$. According to Theorem 4.6(i)[3], $\mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$ qualifies as a h-BCK-I of K.
The S-reflexivity of $\mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$ allows us to invoke Theorem 2.3(i)[3] , which establishes that $\mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$ is a reflexive-h-BCK-I of K. Hence, $\mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$ is a $strong$-h-BCK-I of K.
Since, $\tilde{\zeta}_{\mathfrak{M}}(\hbar) \geq \tilde{t}$ implies $\hbar \in \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$, $(\mathfrak{k} \star \hbar) \cap \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t}) \neq \emptyset$ and $\hbar \in \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$, then $\mathfrak{f} \in \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$ and so $\left(\mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f})\right) \cap \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t}) \neq \emptyset$. The reflexivity of $\mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$ and Theorem 3.5(ii)[3], together imply $\mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f}) \ll \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$. Since $\mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$ is an I-h-BCK-I of K we deduce $\mathfrak{f} \in \mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$, and hence

$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k}) \geq \tilde{t} = \min\left\{\sup_{\mathfrak{k}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{k})\,, \tilde{\zeta}_{\mathfrak{M}}(\hbar)\right\}.$$

With, $\mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$ being an S-$reflexive$-I-h-BCK-Is of K. Let $\mathfrak{f}, \mathfrak{g}, \hbar \in K$. Put

$$\tilde{v} = \max\left\{\inf_{\mathfrak{x}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x})\,, \tilde{\varpi}_{\mathfrak{M}}(\hbar)\right\} \Rightarrow \inf_{\mathfrak{x}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}) \leq \tilde{v} \text{ and } \tilde{\varpi}_{\mathfrak{M}}(\hbar) \leq \tilde{v},$$

The “inf” property of $\tilde{\varpi}_{\mathfrak{M}}$ implies the existence of $\mathfrak{x}_0 \in (\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})$ such that

$$\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}_0) = \inf_{\mathfrak{x}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}) \leq \tilde{v}$$

Therefore $\mathfrak{x}_0 \in \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$, which by $(hBCK2)$, implies
$\left(\left(\mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f})\right) \star \hbar\right) \cap \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v}) = \left((\mathfrak{f} \star \hbar) \star (\mathfrak{g} \star \mathfrak{f})\right) \cap \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v}) \neq \emptyset$, then there exists $\mathfrak{x} \in \left(\mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f})\right)$ such that $(\mathfrak{x} \star \hbar) \cap \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v}) \neq \emptyset$. According to Theorem 4.6(i)[3], $\mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$ qualifies as a h-BCK-I of K. The S-reflexivity of $\mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$ allows us to invoke Theorem 2.3(i)[3] , which establishes that $\mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$ is a reflexive-h-BCK-I of K. Hence, $\mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$ is a $strong$-h-BCK-I of K.
Since, $\varpi_{\mathfrak{M}}(\hbar) \leq v$ implies $\hbar \in \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$, $(\mathfrak{x} \star \hbar) \cap \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v}) \neq \emptyset$ and $\hbar \in \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$, then $\mathfrak{f} \in \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$ and so $\left(\mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f})\right) \cap \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v}) \neq \emptyset$. The reflexivity of $\mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$ and Theorem 3.5(ii)[3], together imply $\mathfrak{f} \star (\mathfrak{g} \star \mathfrak{f}) \ll \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$. Since $\mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$ is an I-h-BCK-I of K we deduce $\mathfrak{f} \in \mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$, and hence

$$\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x}) \leq \tilde{v} = \max\left\{\inf_{\mathfrak{x}\in(\mathfrak{f}\star\hbar)\star(\mathfrak{g}\star\mathfrak{f})} \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{x})\,, \tilde{\varpi}_{\mathfrak{M}}(\hbar)\right\}.$$

Therefore, $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ qualifies as an IVNF-I-h-BCK-I of K.

Theorem 4.7 Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ be an IVNFS on K.

(1) If $\widetilde{\mathfrak{M}}$ satisfies the “sup-inf” property and for all $\forall\ \tilde{s}, \tilde{t}, \tilde{v} \in \mathbb{A}[0,1]$, $\mathcal{U}(\tilde{\xi}_{\mathfrak{M}};\ \tilde{s})$, $\mathcal{U}(\tilde{\zeta}_{\mathfrak{M}};\ \tilde{t})$ and $\mathcal{L}(\tilde{\varpi}_{\mathfrak{M}};\ \tilde{v})$ are $reflexive$ and $\widetilde{\mathfrak{M}}$ is a NF-I-h-BCK-I of K, then $\widetilde{\mathfrak{M}}$ is a NFPI-h-BCK-I of type 3.

(2) Let K be a PI-h-BCK-algebra. If $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ is an IVNF-$weak$-I-h-BCK-I of K, then $\widetilde{\mathfrak{M}}$ is an IVNFPI-h-BCK-I of type 1.

Theorem 4.8 Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ be an IVNFS on K. Then $\widetilde{\mathfrak{M}}$ is an IVNF-$weak$-I-h-BCK-I $\Leftrightarrow$ the IVFSs $\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}$, and $\tilde{\varpi}_{\mathfrak{M}}^{c}$ are F-$weak$-I-h-BCK-Is of K.

Theorem 4.9 Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ be an IVNFS on K. Then $\widetilde{\mathfrak{M}}$ is an IVNF-$weak$-I-h-BCK-I $\Leftrightarrow$ the IVFSs $\diamond\, \mathfrak{M} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\xi}_{\mathfrak{M}}^{c})$, $\circ\, \mathfrak{M} = (\tilde{\zeta}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}^{c})$ and $\triangle\, \mathfrak{M} = (\tilde{\varpi}_{\mathfrak{M}}^{c}, \tilde{\varpi}_{\mathfrak{M}})$ are IVNF-$weak$-I-h-BCK-Is of K.

Proof: The proof of this theorem is analogous to that of Theorem 4.8.

Theorem 4.10 Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ be an IVNFS on K. Then $\widetilde{\mathfrak{M}}$ is an IVNF-I-h-BCK-I ⇔ the IVFSs $\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}$, and $\tilde{\varpi}^{c}_{\mathfrak{M}}$ are IVF-I-h-BCK-Is.

Theorem 4.11 Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ be an IVNFS on K. Then $\widetilde{\mathfrak{M}}$ is an IVNF-I-h-BCK-I ⇔ the $\diamond \mathfrak{M} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\xi}^{c}_{\mathfrak{M}})$, $\circ \mathfrak{M} = (\tilde{\zeta}_{\mathfrak{M}}, \tilde{\zeta}^{c}_{\mathfrak{M}})$ and $\triangle \mathfrak{M} = (\tilde{\varpi}^{c}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ are IVF-I-h-BCK-Is.

V. Conclusion

This research has successfully explored the application of IVNFS to h-BCK-Is within K, providing a significant contribution to the development of NFS-theory in K. The introduction of the concept of IVN-fuzzification of (*strong*, *weak*, *s-weak*) h-BCK-Is has permitted to establish that every IVNF-s-*weak*-h-BCK-I of K is an IVNF-*weak*-h-BCK-I, clarifying new light on the properties and characterizations of IVNF h-BCK-Is. In addition, the definition and characterization of NF-(*weak*)-I-h-BCK-Is of K, as well as the analysis of their relationships with other notions such as NF-(*strong*, *weak*, *reflexive*)-h-BCK-Is and NFPII-h-BCK-ideals of types-1, 2 ...8, have provided valuable insights and related results. The findings of this study have far-reaching implications for the development of NFS-theory and its applications in K, and are expected to inspire further research in this area. Overall, this study has demonstrated the potential of IVNFS to provide a powerful tool for dealing with uncertainty and imprecision in K, and has paved the way for future studies to explore the applications of NFS-theory in a wide range of fields.

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