Industrial Engineering Journal
ISSN: 0970-2555
Volume : 54, Issue 2, No.2, February : 2025

TOPOLOGICAL FRAMEWORKS IN FUZZY STRUCTURES

Dr. Rahul Deo Awasthi Guest Faculty, Department of Mathematics Maharaja Chhatrasal Bundelkhand University Chhatapur Madhya Pradesh

Abstract
In addition to its broad applications across numerous fields of human activity, fuzzy mathematics has undergone extensive theoretical development, creating meaningful links with traditional branches of pure mathematics such as Algebra, Analysis, Geometry, and Topology. This advancement has contributed to a deeper understanding of both classical and modern mathematical frameworks. The current paper delves into the evolution of concepts that facilitated the extension of topological spaces the most comprehensive category of mathematical spaces into fuzzy structures, enriching the field with new perspectives and methodologies.
Fuzzy topological spaces (FTS) and soft topological spaces (STS) are introduced as extensions of classical topology, allowing for more flexible and nuanced interpretations of mathematical properties. Fundamental topological concepts, including limits, continuity, compactness, and Hausdorff spaces, are redefined within the context of these fuzzy and soft structures. This redefinition enables a broader and more adaptable framework for analyzing mathematical problems where uncertainty or imprecision is inherent. The paper also presents practical examples to illustrate how these concepts function within fuzzy and soft topological spaces, highlighting their relevance and applicability in both theoretical and real-world scenarios. These developments not only broaden the scope of topology but also open new avenues for research and application in various scientific and engineering disciplines.

Keywords:
Fuzzy set, soft set, fuzzy topological space (FTS), soft topological space (STS).

1. INTRODUCTION
Since Zadeh introduced fuzzy set theory in 1965 [1], extensive research has been conducted to enhance its ability to handle uncertainty, ambiguity, and vagueness. This has led to numerous extensions and generalizations of the original fuzzy set concept, including interval-valued fuzzy sets, type-2 fuzzy sets, intuitionistic fuzzy sets, and neutrosophic sets. Additionally, alternative mathematical frameworks such as rough sets, soft sets, and grey systems have been proposed to address similar challenges (e.g., see [2]). These advanced mathematical tools have enabled experts to model situations characterized by imprecise or undefined parameters, thereby facilitating the mathematical resolution of problems articulated in natural language without precise numerical data.
Consequently, the scope of applications for fuzzy sets and their related extensions has expanded rapidly, now encompassing nearly all sectors of human activity, including Physical Sciences, Economics, Management, Expert Systems, Industry, Robotics, Decision Making, Programming, Medicine, Biology, Humanities, Human Reasoning, and Education (e.g., see [3, Chapter 6], [4-7]). Beyond practical applications, fuzzy mathematics has also seen significant theoretical development, forging valuable connections with classical branches of pure mathematics such as Algebra, Analysis, Geometry, and Topology.
Topological spaces [8], recognized as the most general category of mathematical spaces where key concepts like limits, continuity, and compactness are defined, have been extended into fuzzy structures. This paper outlines the progression of ideas that facilitated this extension, focusing on the introduction of fuzzy topological spaces (FTS) and soft topological spaces (STS). Examples are provided to illustrate these concepts, with FTSs defined in Section 2, STSs in Section 3, and concluding remarks, along with suggestions for future research, presented in Section 4.

2. FUZZY TOPOLOGICAL SPACES

Industrial Engineering Journal
ISSN: 0970-2555
Volume : 54, Issue 2, No.2, February : 2025

2.1 Fuzzy Sets

The concept of fuzzy sets was first introduced by Zadeh in 1965 [1].

Definition 1: A fuzzy set AAA in a universal set UUU is characterized by its membership function m: $U\rightarrow [0,1]$. It is represented as a set of ordered pairs: $A=\{(x, m(x)): x \in U\}$

Here, m(x) denotes the degree of membership of the element x in A. A higher m(x) indicates a stronger association of xxx with the defining property of A. For simplicity, many researchers equate a fuzzy set directly with its membership function.

However, defining the membership function presents challenges, as it is subjective and varies based on individual perceptions of real-world scenarios. For instance, a person 1.80 m tall might be considered "tall" by one observer but viewed as of average height by another. The only requirement for defining a membership function is adherence to common logic. A fuzzy set fails to accurately represent reality if it violates this, such as assigning a membership degree of ≥ 0.5 in the "tall people" set to individuals shorter than 1.50 m.

A classical (crisp) subset A of U can be considered a fuzzy set where the membership function is defined as m(x)=1m(x) = if x ∈ A x and m(x) = 0 otherwise. The fundamental properties of crisp sets extend naturally to fuzzy sets [3].

Definition 2: The universal fuzzy set F_U and the empty fuzzy set $F_\emptyset$ in U are defined by their membership functions m(x)= 1 and m(x) = 0 for all x ∈ U, respectively. It can be readily verified that for any fuzzy set A in U:

- $A\cup F_U = F_U$
- $A\cap F_U = A$
- $A\cup F_\emptyset = A$
- $A\cap F_\emptyset = F_\emptyset$

Definition 3: Let A and B be fuzzy sets in U with membership functions m_A and m_B respectively. A is a fuzzy subset of B if $m_A(x) \leq m_B(x)$ for all $x\in U$, denoted $A\subseteq B$. If $m_A(x) < m_B(x)$ for all $x\in U$, A is a proper fuzzy subset of B, denoted $A\subset B$.

Definition 4: For fuzzy sets A and B in U with membership functions m_A and m_B:

- The **union** $A\cup B$ is a fuzzy set in U with membership function

$m_{A\cup B(x)} = \max\{m_A(x), m_B(x)\}$.

- The **intersection** $A\cap B$ is a fuzzy set in U with membership function

$m_{A\cap B(x)} = \min\{m_A(x), m_B(x)\}$

- The **complement** of A, denoted A^C, is a fuzzy set in U with membership function

${m_A}^C{}_{(x)} = 1 - m_A(x)$

Example 1:

Let U represent a set of human ages:

U={5,10,20,30,40,50,60,70,80

Table 1 provides the membership degrees of these elements in the fuzzy sets A= young, B=adult and C=old.

1. Prove that $C\subseteq B^C = C\subset B^C$?
2. Calculate the fuzzy sets $A^C \cap C$ and $(A^C\cup B) \cap C$

Table 1: Human age

U	A	B	C
A			
B			
C			
U			
A			
B			
C			

Industrial Engineering Journal
ISSN: 0970-2555
Volume : 54, Issue 2, No.2, February : 2025

U			
5	1	0	0
10	1	0	0
20	0.8	0.8	0
30	0.5	1	0.2
40	0.2	1	0.4
50	0.1	1	0.6
60	0	1	0.8
70	0	1	1
80	0	1	1

Solution: 1) From Table 1 turns out that $m_C(x)$ □□□B(x), □ x □□U. Thus C □□B. But m_C (70) □□m_B(70)□1,therefore it is not true that C □□B

2) Calculating min□$m_A(x)$,$m_C(x)$□,□x□U one finds that A∩C={(5,0),(10,0),(20,0),(30,0.2),(40,0.2),(50,0.1),(60,0),(70,0),(80,0)}.

Also, calculating max□$m_{A□C}(x)$,$m_B(x)$□,∀x∈ U, one finds that (A□C)□B={(5,0),(10,0),(20,0),(30,0.2),(40,1),(50,1),(60,1),(70,1),(80,1)}.

Fuzzy Topologies

Definition 5 [9]: A *fuzzy topology (FT)* T on a non- empty set U is a collection of FSs in U such that:

- The universal and the empty FSs belong to T
- The intersection of any two elements of T and the union of an arbitrary (finite or infinite) number of elements of T belong also to T.

Examples 2: Trivial examples of FTs are the *discrete FT* {F∅, FU} and the *non-discrete FT* consisting of all FSs in U. Another example is the collection of all *constant FSs* in U, i.e .all FSs in U with membership function of the form m(x)=c, for some c in [0, 1] and all x in U.

The elements of a FT T on U are called *open fuzzy sets* in U and their complements are called *closed fuzzy sets* in U. The pair (U, T) defines a *fuzzy topological space (FTS)* on U.

Next we describe how one can extend the concepts of *limit, continuity, compactness, and Hausdorff space* to FTSs [9].

Definition 6: Given two fuzzy sets A and B of the FTS (U, T), B is said to be a *neighborhood* of A, if there exists an open fuzzy set O such that A□O□B.

Definition 7: We say that a sequence {A_n} of fuzzy sets of (U,T) converges to the fuzzy set A of (U,T), if there exists a positive integer m, such that for each integer n≥m and each neighborhood B of A we have that A_n□B. Then A is called the *limit* of {A_n}.

Lemma1: (*Zadeh's extension principle*) Let X and Y be two non-empty crisp sets and let f: X□Y be a function. Then f can be extended to a function F mapping FSs in X to fuzzy sets in Y.

Proof: Let A be a fuzzy set in X with membership function m_A. Then its image F(A) is the fuzzy set B in Y with membership function m_B, which is defined as follows: Given y in Y, consider the set$f^{-1}(y)$={x ∈X: f(x) = y}. If $f^{-1}(y)$=∅, then $m_B(y)$=0, and if $f^{-1}(y)$≠∅,then $m_B(y)$ is equal to the maximal value of all $m_A(x)$ such that x ∈ $f^{-1}(y)$. Conversely, the inverse image $F^{-1}(B)$ is the fuzzy set A in X with membership function $m_A(x)=m_B(f(x))$, for each x ∈ X.

Definition 8: Let (X,T) and (Y,S) be two FTSs and let f: X□Y be a function. By Lemma1,f can be extended to a function F mapping fuzzy sets in X to fuzzy sets in Y. We say then that f is a *fuzzy continuous* function, if, and only if, the inverse image of each open fuzzy set in Y through F is an open fuzzy set in X.

Definition 9: A family A={A_i, i∈ I}of fuzzy sets of a FTS (U,T) is called a *cover* of U, if U=⋃A_i.If

2.A *T2-FTS* (or a *separable* or a *Hausdorff FTS*), if, and only if, for each pair of elements u_1, u_2 of U, $u_1 \neq u_2$, there exist at least two open fuzzy sets O_1 and O_2 such that $u_1 \in O_1$, $u_2 \in O_2$ and $O_1 \cap O_2 = \emptyset_F$. Obviously a T_2-FTS is always a T_1-FTS.

Soft Topological Spaces

Soft Sets

The need of passing through the existing difficulty to define properly the membership function of a fuzzy set gave the hint to D. Molodstov, Professor of Mathematics at the Russian Academy of Sciences, to introduce in 1999 the concept of *soft set* [10] as a tool to tackle the existing in real world uncertainty in a parametric manner. A soft set is defined as follows:

Definition 10: Let E be a set of parameters, let Abe a subset of E, and let f be a map from A into the power set P(U) of all subsets of the universe U. Then the soft set (f, A) in U is defined to be the set of the ordered pairs

$(f,A) = \{(e,f(e)): e \in A\}$(2)

In other words, a soft set in U is a parameterized family of subsets of U. The name "soft" was given because the form of (f, A) depends on the parameters of A. For each e ∈ A, its image f(e) is called the *value set* of e in (f, A), while f is called the *approximation function* of (f, A).

Example 3: Let U= $\{C_1, C_2, C_3\}$ be a set of cars and let E = $\{e_1, e_2, e_3\}$ be the set of the parameters e_1=cheap, e_2=hybrid (petrol and electric power) and e_3= expensive. Let us further assume that C_1, C_2 are cheap, $C_3$is expensive and C_2, $C_3$are the hybrid cars. Then, a map f: E□P(U) is defined by $f(e_1)=\{C_1,C_2\}$,$f(e_2)=\{C_2,C_3\}$and $f(e_3)=\{C_3\}$.

Therefore, the soft set (f, E) in U is the set of the ordered pairs (f, E) = {(e_1, $\{C_1, C_2\}$), (e_2, $\{C_2, C_3\}$, (e_3, $\{C_3\}$}.

A fuzzy set in U with membership function y=m(x) is a soft set in U of the form (f, [0, 1]), where the elements of A are open fuzzy sets, then A is called an *open cover* of U. Also, each subset of A being also a cover of U is called a *sub-cover* of A. The FTS (U, T) is said to be *compact*, if every open cover of U contains a sub-cover with finitely many elements.

Definition11: AFTS (U,T) is said to be:

1. A *T1-FTS*, if, and only if, for each pair of elements u_1, $u_2$of U, $u_1 \neq u_2$, there exist at least two open fuzzy sets $O_1$and O_2such that $u_1 \in O_1$, u_2□$O_1$and $u_2 \in O_2$, u_1□O_2.

f(α)={x□U:m(x)□α}is the corresponding *α–cut* of the fuzzy set, for each α in [0,1].

It is of worth noting that, apart from soft sets, which overpass the existing difficulty of defining properly membership functions through the use of the parameters of E, alternative theories for managing the uncertainty have been also developed, where the definition of a membership function is either not necessary (*grey systems/numbers* [11]), or it is over passed by using a pair of crisp sets which give the lower and the upper approximation of the original crisp set (*rough sets* [12]).

The basic definitions on soft sets are introduced in a way analogous to fuzzy sets (see section 2.1)

Definition 12: The *absolute soft set* A_U is defined to be the soft set (f, A) such that f(e)=U, ∀ e ∈ A, and the *null soft set* $A_\emptyset$ is defined to be the soft set (f, A) such f(e)=∅, ∀ e ∈A.

It is straight forward to check that for each soft set A in U is $A \cup A_U = A_U$, $A \cap A_U = A$, $A \cup A_\emptyset = A$ and $A \cap A_\emptyset = A_\emptyset$.

Definition13: If (f,A) and (g,B) are two soft sets In U,(f,A) is called as *of t subset* of (g,B), if A□B And f (e)□g(e), ∀e ∈A. We write then (f, A)□(g, B). If A□B, then (f, A) is called a *proper soft subset* of B and we write (f, A)□(g, B).

Definition 14: Let (f, A) and (g, B) be two soft sets in U. Then:

- The *union* (f,A)∪ (g,B) is the SS (h, A∪B) in U, with h(e)=f(e) if e ∈ A-B, h(e)=g(e) if e ∈ B-A and h(e)= f(e)∪g(e) if e ∈ A∩B.
- The *intersection* (f, A) ∩ (g, B) is the soft set(h, A∩B) in U, with (e)=f(e)∩g(e),∀e

Industrial Engineering Journal
ISSN: 0970-2555
Volume : 54, Issue 2, No.2, February : 2025
∈ A∩B.

- The *complement* $(f, A)^C$ of the soft SS(f, A) in U, is the SS (f^C, A) in U, for which the function f^C is defined by $f^C(e)$ = U-f(e), ∀ e
∈A.

For general facts on soft sets we refer to [13].

Example 4: Let U={H1, H2, H3}, E={e1, e2, e3} and A= {e1, e2}**.**Consider the soft set S=(f,A)={(e1,{H1,H2}),(e2,{H2,H3})}in U.

Then the soft subsets of S are the following: S1={(e1,{H1})},S2={(e1,{H2})},S3={(e1,{H1,

Definition15: A *soft topology* T on a non-empty set U is a collection of SSs in U with respect to a set of parameters E such that:

- The absolute and the null soft sets EUand E∅belong to T
- The intersection of any two elements of T and the union of an arbitrary (finite or infinite) number of elements of T belong also to T.

The elements of a ST T on U are called *open SS* and their complements are called *closed SS*. The triple (U, T, E) is called a STS on U.

Examples 5: Trivial examples of STs are the *discrete ST* {E∅, EU} and the *non-discrete ST* consisting of all SSs in U. Reconsider also Example
4. It is straight forward to check then that *T={EU,*
E∅, S, S2, S9, S11}is a ST on U.

The concepts of limit, continuity, compactness, and Hausdorff TS are extended to STSs in a way analogous of FTSs [19, 20]. In fact, Definitions 7, 9 and 10 are easily turned to corresponding definitions of STSs by replacing the expression “fuzzy sets” with the expression “soft sets”. For the concept of continuity we need the following Lemma ([19], definition 3.12):

Lemma 2: Let (U, T. A), (V, S, B) be STSs and let u: U→V, p: A→B be given maps. Then a map fpu is defined with respect to u and p mapping the soft sets of T to soft sets of S.

Proof: If (F, A) is a soft set of T, then its image fpu ((F, A)) is a soft set of S defined by
fpu ((F,A))=(fpu(F),p(A)), where, ∀y∈ B is
H2})}, S4= {(e2,{H2})},S5={(e2,{H3})},S6={(e2,
{H2, H3})},S7={(e1,{H1},(e2,{H2})},S8={(e1,
{H},(e,{H})},S={(e,{H},(e,{H})},
fpu (F)(y)=
ifp^{-1}(y)∩A≠∅and
f(F)(y) = ∅ otherwise.
S10={(e1,{H2},(e2,{H3})},S11={(e1,{H1,H2},(e2,
{H2})},S12={(e1,{H1,H2},(e2,{H3})},S13={(e1,
{H1},(e2,{H2,H3})},S14={(e1,{H2},(e2,{H2, pu

Conversely, if (G,B) is a soft set of S, then its inverse image f^{-1}((G,B)) is a soft set of T defined By
f^{-1} ((G,B))=(f^{-1}(G),p^{-1}(B)), where ∀ x∈A is pu
H3})}, S**,** A ∅={(e1,∅),(e2,∅)}
It is also easy to check that $(f,A)^C$={(e1,{H3}),(e2,
{H1})}**.**

3.3 Soft Topologies

Observe that the concept of FTS (Definition 5) is obtained from the classical definition of TS [8] by replacing in it the expression “a collection of subsets of U” by the expression “a collection of FSs in U”. In an analogous way one can define the concepts of *in tuitionistic FTS (IFTS)* [14], of *neutrosophic TS (NTS)* [15, 16], of *rough TS (RTS)* [17], of *soft TS (STS)* [18], etc. In particular, a STS is defined as follows:

Industrial Engineering Journal
ISSN: 0970-2555
Volume : 54, Issue 2, No.2, February : 2025

pu pu
f^{-1} (G)(x)=u^{-1}(G(p(x)). pu

Definition 16: Let (U, T. A), (V, S, B) be STSs and let u: U→V, p: A→B be given maps. Then the map fpu, defined by Lemma 2, is said to be *soft pu- continuous*, if, and only if, the inverse image of each open soft set in Y through fpu is an open soft set inX.

4. Discussion and Conclusions

In this review, we explored the concepts of Fuzzy Topological Spaces (FTS) and Soft Topological Spaces (STS), extending fundamental mathematical notions such as limits, continuity, compactness, and Hausdorff spaces to these frameworks.

Beyond its wide-ranging practical applications, fuzzy mathematics has also undergone substantial theoretical advancements. These developments have established valuable connections with traditional areas of pure mathematics, including Algebra, Analysis, Geometry, and Topology. Given its theoretical depth and interdisciplinary relevance, this field presents promising opportunities for future research.

References:

1. Zadeh, L.A., *Fuzzy Sets*, *Information and Control*, 8, 1965, pp. 338-353.
2. Voskoglou, M.Gr., *Fuzzy Systems, Extensions and Relative Theories*, *WSEAS Transactions on Advances in Engineering Education*, 16, 2019, pp. 63-69.
3. Klir, G.J. & Folger, T.A., *Fuzzy Sets, Uncertainty and Information*, Prentice-Hall, London, 1988.
4. Deng, J., *Introduction to Grey System Theory*, *The Journal of Grey System*, 1, 1989, pp. 1-24.
5. Kharal, A. & Ahmad, B., *Mappings on Soft Classes*, *New Mathematics and Natural Computation*, 7(3), 2011, pp. 471-481.
6. Voskoglou, M.Gr., *Finite Markov Chain and Fuzzy Logic Assessment Models: Emerging Research and Opportunities*, CreateSpace Independent Publishing Platform, Amazon, Columbia, SC, USA, 2017.
7. Atanassov, K.T., *Intuitionistic Fuzzy Sets*, Physica-Verlag, Heidelberg, N.Y., 1999.
8. Willard, S., *General Topology*, Dover Publications Inc., N.Y., 2004.
9. Chang, S.L., *Fuzzy Topological Spaces*, *Journal of Mathematical Analysis and Applications*, 24(1), 1968, pp. 182-190.
10. Molodtsov, D., *Soft Set Theory—First Results*, *Computers and Mathematics with Applications*, 37(4-5), 1999, pp. 19-31.
11. Deng, J., *Control Problems of Grey Systems*, *Systems and Control Letters*, 1982, pp. 288-294.
12. Pawlak, Z., *Rough Sets: Aspects of Reasoning about Data*, Kluwer Academic Publishers, Dordrecht, 1991.
13. Luplanlez, F.G., *On Intuitionistic Fuzzy Topological Spaces*, *Kybernetes*, 35(5), 2006, pp. 743-747.
14. Salama, A.A. & Alblowi, S.A., *Neutrosophic Sets and Neutrosophic Topological Spaces*, *IOSR Journal of Mathematics*, 3(4), 2013, pp. 31-35.
15. Voskoglou, M.Gr., *Managing the Existing in Real Life Indeterminacy*, *International Journal of Mathematical and Computational Methods*, 7, 2022, pp. 29-34.
16. Lashin, E.F., Kozae, A.M., Abo Khadra, A.A. & Medhat, T., *Rough Set for Topological Spaces*, *International Journal of Approximate Reasoning*, 40, 2005, pp. 35-43.
17. Shabir, M. & Naz, M., *On Soft Topological Spaces*, *Computers and Mathematics with Applications*, 61, 2011, pp. 1786-1799.
18. Zorlutuna, I., Akdag, M., Min, W.K. & Amaca, S., *Remarks on Soft Topological Spaces*, *Annals of Fuzzy Mathematics and Informatics*, 3(2), 2011, pp. 171-185.
19. Georgiou, D.N., Megaritis, A.C. & Petropoulos, V.I., *On Soft Topological Spaces*, *Applied Mathematics & Information Sciences*, 7(5), 2013, pp. 1889-1901.