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CLUSTER DECAY OF EVEN-ODD AND ODD-ODD SUPER HEAVY NUCLEI

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***Abstract*—**Half-life of a cluster is a very important parameter to identify it in a nuclear reaction where it gets deposited and detected in the detection system. Half-life is also very important parameter to understand physical properties of nucleus. Here we use the energy eigenvalue as a complex eigenvalue to determine the decay width from which we can calculate the half-life. To calculate the half-life of a cluster we assume it to be a point particle. Here we develop the method to calculate the decay width and half-life. From this we show that even for the super heavy nuclei region the dominant decay process is the alpha decay process over the cluster decay process.

Keywords*—Cluster Decay, Super Heavy Nuclei,*

I .INTRODUCTION

The stability of Super- Heavy Nuclei can be greatly studied by the shell model. Super heavy elements do not have perfectly spherical nuclei. A spherical nucleus is considered to be most stable. Research studies proved that large nuclei are deformed causing the magic number to shift. [1] Since Super Heavy Nuclei are unstable these nuclei are artificially created to study its properties. The half-lives of different radioactive decays such as alpha decays, cluster decays and spontaneous fission are signs of formation of Super Heavy Nuclei. [2] The Super Heavy Nuclei decays through the two principle decay modes one is alpha decay and the other is spontaneous fission. There are various approaches for calculation of half-lives like coulomb and proximity shell model, Cluster models, semi classical approach of WKB approximations and also the viola-seaborg formula which is the phenomenological formula for alpha decay half-lives from experimental Q values. In the WKB approach the transition probability is calculated and to find the half-life and decay width the probability is multiplied by the assault frequency (a classical parameter). [3] Hence it is not a fully quantum mechanical approach. In the present work we will obtain the decay width from full quantum mechanical treatment. Before going to the procedure for finding the decay width we will first familiarize ourselves with the idea of a complex energy state. Such a state does not belong to the Hilbert space. We will develop the understanding of complex energy and such energy states in detail. By considering the state as a gamow state we will get decay width directly from which we will get the half-lives. Nowadays research on Super Heavy Nuclei is done extensively to investigate the existence of the upper bound of the periodic table.[4]Another curiosity is whether the magic number exists in Super Heavy Nuclei, or why do they fission out as soon as they are formed. The nuclear stability is determined by the interaction of nucleons inside the nucleus. Stability is decreased as we set for the large nuclear mass which is shown by the probability of spontaneous fission.

II. REVIEW

In recent research it has been seen that if we generalize the real energy eigenvalue to the complex energy eigenvalue it gives a rich amount of information about the system like if we consider the complex part of the energy it shows decay width of the decaying system. Let us see how it shows the decay width. Consider the decaying system. The outgoing particle has a free particle solution at a large distance; this wave function will spread out as time passes, thus the probability density at any point will tend to zero as time tends to infinity. Hence it is not a stationary state and breaks the time independence. But for continuum states like above, decaying states have a norm which is equal to infinite and therefore such states do not lie inside usual Hilbert space. To define this state one has to construct a space which is a superset of Hilbert space. The Resonance state or decay state is one of such states. Such states are greatly describe by the Gamow states which can be shown by eigenvector $|E_n - i\Gamma/2\rangle$. Hermiticity of the Hamiltonian is broken due to the outgoing boundary condition therefore it has complex energy eigenvalue. Let us see how these states describe the decay state or resonance state. Consider time evolution of wave function

$$\Phi_n(r,t) = \Phi(r)e^{-i\varepsilon_n t/\hbar} \quad (1)$$

where, $\varepsilon_n = E_n - i\Gamma/2$, therefore

$$\Phi_n(r,t) = \Phi(r)e^{-iE_n t/\hbar}e^{-\Gamma t/\hbar} \quad (2)$$

Assume that Φ_n is properly normalized then no. of particles $N_n(t)$ in the state n contained in a one dimensional box of length a is

$$N_n(t) = \int_0^a \quad |\Phi_n(x,t)|^2 dx \quad (3)$$

$$N_n(t) = e^{-\Gamma t/\hbar} \int_0^a \quad |\Phi_n(x)|^2 dx \quad (4)$$

therefore,

$$N_n(t) = e^{-\Gamma t/\hbar} N_n(0) \quad (5)$$

Hence if we choose $\varepsilon = E - i\Gamma/2$, it represents a resonance. In our barrier penetration we don't have a bound state since the potential is either zero or repulsive. But potentials like given in fig.1 have both bound and resonance state. The real part shows the position and imaginary part shows half the width of resonance. From equation the mean lifetime of the system is the time T_n at which the number of particles in the box has diminished by e i.e. $N_n(T_n) = N_n(0)/e$. Therefore

$$N_n(T_n) = \frac{N_n(0)}{e} = e^{-\frac{\Gamma_n T_n}{\hbar}} N_n(0) \quad (6)$$

$$\frac{1}{e} = e^{-\frac{\Gamma_n T_n}{\hbar}} \quad (7)$$

$$\Gamma_n T_n = \hbar \quad (8)$$

The above relation is known as a gamow relation. In Fig.1 the spectrum of the system is represented. One can see that resonance has a width which is minus twice the imaginary part of the energy indicated by the red lines. From the gamow relation we see that width reflects the time in which the system stays in the resonance states. Wider the resonance i.e. Γ_n, smaller would be the time T_n at which the system is trapped inside the barrier. At zero energy the width is small since the barrier is high and the mean time at which the particle stays inside the barrier is large. At the top of the barrier resonance is wide hence the particle can easily escape, therefore mean time is short. If the height of the barrier tends to infinity all states have a zero width i.e. all states will be bound as we see in the infinite potential well. Therefore Gamow resonances are generalizations of bound states. Gamow resonance and bound states have a common property of obeying outgoing boundary conditions.

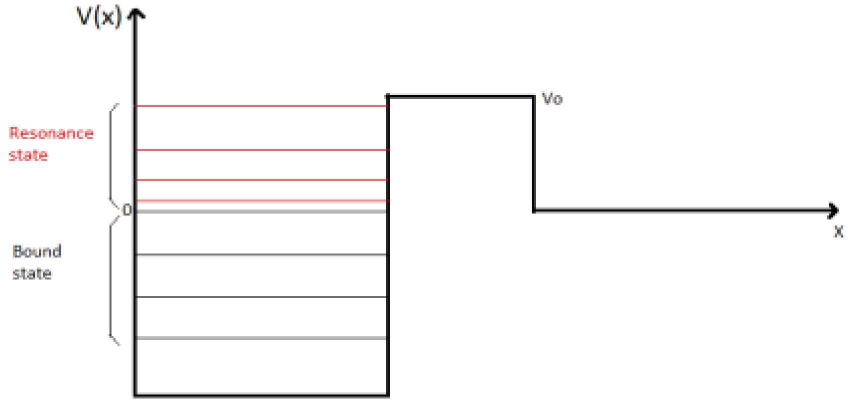

Fig1. Semi-infinite potential well.

Therefore we will call all states satisfying outgoing boundary conditions as Gamow states. These Gamow states are bound states as well Gamow resonances. In the complex K-plane we will call $K = \kappa + i\gamma$

And

$$\varepsilon = \frac{\hbar^2 K^2}{2m} = \frac{\hbar^2}{2m}(\kappa^2 - \gamma^2 + 2i\kappa\gamma) \qquad (9)$$

With $\varepsilon = E - i\Gamma/2$

one gets

$$E = \frac{\hbar^2}{2m}(\kappa^2 - \gamma^2) \qquad (10)$$

$$\Gamma = -\frac{2\hbar^2}{m}\kappa\gamma \qquad (11)$$

The wave function behaves at large distance as

$$u(K, r) \to Ne^{iKr} = Ne^{i\kappa r}e^{-\gamma r}$$

(12)

and therefore it is outgoing (incoming) if $\kappa > 0$ ($\kappa < 0$).

There are four classes of poles as shown in fig.2. They are

1) Bound states, $\kappa = 0, \gamma > 0$,

$$E = -\frac{\hbar}{2m}\gamma^2 < 0, \Gamma = 0, u(K, r) \to e^{-\gamma r} \to 0 \text{ converges,}$$

2) Anti-bound states, $\kappa = 0, \gamma < 0$,

$$E = -\frac{\hbar}{2m}\gamma^2 < 0, \Gamma = 0, u(K, r) \to e^{|\gamma| r} \to \infty \text{ diverges,}$$

3) Decaying states, $\kappa < 0, \gamma > 0$,

$$E = \frac{\hbar}{2m}(\kappa^2 - \gamma^2) < 0, \Gamma = -\frac{2\hbar}{m}\kappa|\gamma| < 0, u(K, r) \to e^{i\kappa r}e^{|\gamma| r} \to \infty \text{ diverges,}$$

4) Capturing states, $\kappa < 0, \gamma < 0$,

$$E = \frac{\hbar}{2m}(\kappa^2 - \gamma^2) < 0, \Gamma = -\frac{2\hbar}{m}|\kappa||\gamma| < 0, u(K, r) \to e^{i\kappa r}e^{|\gamma| r} \to \infty \text{ diverges,}$$

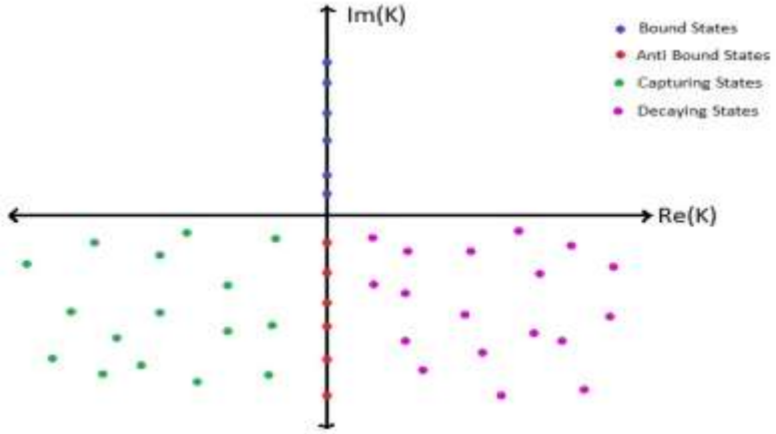

Fig 2. Complex plane representing Bound states, Anti Bound states, Capturing States and Decaying states.

Notice, for all class of states Γ is zero or negative. The above diverging wave function is normalized by an appropriate renormalization. The anti-bound states which are not real (virtual) states well about the formation of halo nuclei.

III. FORMALISM

Here cluster is considered to be point particle. [6] It is considered that a cluster moving inside the well tries to penetrate the barrier. The smallest clusters emitted are alpha particles so here we would first give a description for alpha decay which can then be applied to heavy cluster decay. We describe cluster as a point particle with well-defined mass and angular momentum moving in a gamow state under the interaction potential formed by interaction between cluster and daughter nucleus which we assumed are performed in complex state of parent nucleus and hence the size of this potential is considered to be equal to

$$r_0 A^{1/3} \qquad (13)$$

Where, A is the mass of the parent nucleus Here we consider the cluster as a point particle because constructing a cluster wave function with internal structure is a very difficult task. With this assumption of a point particle our model turns out to be a simple effective model. We know super heavy elements usually do not have perfectly spherical nuclei but are deformed in their ground state. Now this deformation depends on how many particles does this nuclei have and how far they are from the shell closure. If the system is spin saturated it will always be spherically symmetric. But as we go away from the shell closure the system gets deformed since not all the orbitals are occupied. So for simplicity we consider even-even cluster emitters. Now even if the cluster is deformed it hides out in our assumption of a point particle. In the present model we consider that a cluster moving inside the well tries to penetrate the barrier and once it is through, the only interaction it has will be purely coulombic. Hence the outgoing wave function is coulomb wave function therefore the radial wave function corresponding to outgoing cluster can be given by.[5]

$$\Psi_{lj}{}^{out}(r) = N_{lj}[G_l(r) + iF_l(r)] \qquad (14)$$

Where, F and G are regular and irregular coulomb functions respectively. The reason for defining wave function as r times function is to get dimension of N^2 as inverse length. Since in this case we are only considering 's' state which corresponds to $l = 0$ and $j = 0$ therefore radial wave function corresponding to outgoing cluster becomes

$$r\,\Psi_{00}{}^{out}(r) = N_{00}[G_0(r) + iF_0(r)] \qquad (15)$$

Here N is a normalization constant and for simplicity let's drop the subscript 00.
Internal wave function for cluster is given by gamow wave function $\phi(r)$ .Since at large distance from the parent nucleus outgoing spherical wave functions become plane waves.

Therefore at limit $r \to \infty$,

$$\lim_{r\to\infty} |r\Psi^{out}(r)|^2 = N^2 \quad (16)$$

The normalization constant is extracted by a matching condition, where both internal and outgoing functions are equal to each other. Now this condition is found at boundary $r = R$ where the outgoing coulomb wave function $\Psi(r)$ is equal to the internal gamow wave function $\phi(r)$

$$R^2|\phi(R)|^2 = R^2|\Psi(R)|^2 \quad (17)$$

$$|N|^2[F^2(kR) + G^2(kR)] = R^2|\Psi(R)|^2 \quad (18)$$

$$|N|^2 = \frac{R^2|\Psi(R)|^2}{[F^2(kR)+G^2(kR)} \quad (19)$$

Here F and G are explicitly shown in terms of kR as coulomb wave function is a function of r and if we look at this function as a power series we cannot add length to length square. We need to make it dimensionless we need to multiply it with the quantity having dimension of inverse length that is k where k is a wave number of the outgoing particle. Now dimensionally matching the internal gamow and outgoing coulomb wave function the quantity N^2 physically turns out to decay probability per unit length at infinity. If we multiply this with v we will get the probability of decay per second. Here v is the velocity of the outgoing particle, and is given by

$$v = \frac{\hbar k}{\mu} \quad (20)$$

Where μ is reduced mass of binary system formed by cluster and daughter and $\hbar k$ momentum of outgoing particle. And

$$v|N|^2 = \frac{\hbar k}{\mu}\frac{R^2|\Psi(R)|^2}{[F^2(kR)+G^2(kR)} \quad (21)$$

Now the quantity decay probability per unit second is 1 over mean lifetime τ. now since we know

$$\tau_{1/2} = ln2\tau \quad (22)$$

We deduce decay half-life as

$$\frac{1}{t_{1/2}} = \frac{1}{ln2\tau} = \frac{1}{ln2}v|N|^2 = \frac{1}{ln2}\frac{\hbar k}{\mu}\frac{R^2|\Psi(R)|^2}{[F^2(kR)+G^2(kR)} \quad (23)$$

than we know that decay width Γ is related to half-life by gamow relation as follows

$$\Gamma = \frac{\hbar}{t_{1/2}} \quad (24)$$

Now by substituting the value for $t_{1/2}$, we get

$$\Gamma = \frac{1}{ln2}\frac{\hbar^2 k}{\mu}\frac{R^2|\Psi(R)|^2}{[F^2(kR)+G^2(kR)} \quad (25)$$

This is how we got the decay width and decay half-life by exact quantum mechanical treatment [5] As the above calculation done for decay in non-deformed even-even nuclei decay in deformed and odd nuclei is also important. So in General, the single particle or point cluster outgoing wave function is given as

$$r\Psi_{lj}{}^{out}(r) = N_{lj}[G_{lj}(r) + iF_{lj}(r)] \quad (26)$$

The probability per second that the particle passes through a area element $dS = r^2 d\theta d\varphi$ is given by $F_{lj} = |\Psi^{out}(r)|^2 2vdS$ where v is the velocity of the particle. Since

$$\lim_{r\to\infty} |r\Psi_{lj}{}^{out}|^2 = |N_{lj}|^2 \quad (27)$$

the probability of decay per second, i.e., the reciprocal of the half-life obtained by integrating F_{lj} about the angles and therefore the decay width is given as

$$\Gamma_{lj} = \frac{1}{ln2}\frac{\hbar^2 k}{\mu}\frac{R^2|\Psi_{lj}(R)|^2}{[F_{lj}{}^2(kR)+G_{lj}{}^2(kR)} \quad (28)$$

which is independent of R. The above expression gives the exact quantum mechanical value of the width.

For our present calculations we will calculate the partial decay width for angular momentum projected states. We will assume that the deformation of the parent nuclei and mother nuclei to be the same also we will assume that the mother nucleus is odd and the wave function for such nucleus is given as [5]

$$\Psi_m{}^{J_iM_iK_i} = \left(\frac{2J_i+1}{16\pi^2}\right)^{1/2} [D_{M_iK_i}{}^{J_I}\chi_{K_i} + (-1)^{J_i+M_i}D_{M_i-K_i}{}^{J_I}\chi_{\check{K}_i}] \quad (29)$$

where D are the Wigner matrices and χ_K is the intrinsic single particle or cluster (which is assumed to be a point particle) wave function which can be expanded in spherical component as

$$\chi_{K_i}(r) = \sum_{j\geq K_i} \alpha_{lj}(r)[Y_l(r)\chi_{1/2}]_{jK_i} \quad (30)$$

where l is determined by the parity of the state.We assume that the decay is possible with maximum probability when the nucleus is in the lowest energy state with $J_i = K_i$.The daughter wave function is given as

$$\Psi_d{}^{J_dM_dK_d} = \left(\frac{2J_d+1}{8\pi^2}\right)^{1/2} D_{M_dK_d}{}^{J_d} \quad (31)$$

The outgoing particle has a maximum energy when it leaves the daughter nucleus in the ground state on which the condition $J_d = K_d = M_d = 0$ can be imposed.Therefore from the conservation of angular momentum the outgoing particle have the same angular momentum as that of parent nucleus i.e. $j_p = J_i = K_i$. At a large distance R there is only coulombic interaction and nuclear interaction vanishes therefore the outgoing particle wave function is given as

$$R\chi_{K_i{}^{out}}(R) = \sum_{lj} N_{lj}[G_{lj}(R) + iF_{lj}(R)][Y_l(R)\chi_{1/2}]_{jK_i} \quad (32)$$

N_{lj} is determined by

$$|R\chi(R)|^2 = |R\chi^{out}(R)|^2 \quad (33)$$

at matching radius R. From the orthogonal condition of the different partial waves we find that the partial decay width of particular decay channel l_pj_p is given by

$$\Gamma_{l_pj_p} = \frac{1}{ln2}\frac{\hbar^2 k}{\mu}\frac{R^2\alpha_{l_pj_p}{}^2(R)}{[F_{l_pj_p}{}^2(kR)+G_{l_pj_p}{}^2(kR)} \quad (34)$$

The cluster is formed just outside the surface of the daughter nucleus. [7]

The formation probability of cluster is the absolute square of the wave function describing the cluster. Therefore the formation probability of cluster is given as

$$P = \int r^2dr|\phi(r)|^2 \quad (35)$$

where $\phi(r)$ is a Gamow wave function. We will show that the above formalism to be successful in calculating the half-lives of alpha particles as well as clusters.

V. CALCULATIONS

We used the computer code GAMOW to evaluate the outgoing cluster wave functions. Here we have assumed the interaction of standard wood saxon form which have three adjustable parameters. [8]

1) Depth (V_0)
2) Half density radius ($R_{1/2} = r_0A^{1/3}$)
3) Diffusivity (a)

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For Depth V_0
In our model the cluster is considered to be a point particle moving in an energy state with energy equal to Q value. So here we fix Q value and adjust the depth. It is observed that it is good to search for the depth around -200 MeV. This value of the Depth is used in all the calculation. [8]

For r_0
It has been found that the value of r_0 has a dependence on the Q value. So we choose r_0 = 1.31 fm for Q value about 70 Mev and r_0 = 1.35 for Q value of about 70 Mev for simplicity. [8]

For a
Considering that the half lives depend on the width of the barrier it is found that diffusivity does not change the barrier width much. Thus on concluding that half-lives are not much dependent to 'a' we set the value of a = 0.54 fm. [8]

VI. RESULTS

Parent	Daughter	Cluster	Q (MeV)	$log_{10}T_{1/2}$	Formation Probability	Q_α (MeV)	$log_{10}T_{1/2}$ (α)	Foramtion Probability(α)
^{252}No	^{230}U	^{22}Ne	59.280	38.263420	4.62338 x 10^{-24}	8.550	-1.176994	9.75500 x 10^{-3}
^{253}No	^{231}U	^{22}Ne	58.580	39.421885	3.47173 x 10^{-24}	8.420	-0.772204	9.54652 x 10^{-3}
^{254}No	^{232}U	^{22}Ne	58.130	40.151607	2.91166 x 10^{-24}	8.230	-0.154401	9.25457 x 10^{-3}
^{255}No	^{233}U	^{22}Ne	57.920	40.453777	2.73227 x 10^{-24}	8.440	-0.870587	9.56535 x 10^{-3}
^{256}No	^{234}U	^{22}Ne	57.700	40.776850	2.55711 x 10^{-14}	8.589	-1.347932	9.77836 x 10^{-3}
^{257}No	^{235}U	^{22}Ne	57.360	41.323732	2.25311 x 10^{-24}	8.480	-1.033477	9.61621 x 10^{-3}

Table 1

Parent	Daughter	Cluster	Q (MeV)	$log_{10}T_{1/2}$	Formation Probability	Q_α (MeV)	$log_{10}T_{1/2}$ (α)	Foramtion Probability(α)
^{246}Fm	^{226}U	^{20}O	39.060	47.223410	6.64312 x 10^{-19}	8.380	-1.300278	0.104272 x 10^{-1}
^{248}Fm	^{228}U	^{20}O	38.880	47.555709	6.35644 x 10^{-19}	8.000	-0.058562	0.097921 x 10^{-1}
^{249}Fm	^{229}U	^{20}O	38.510	48.456228	5.49020 x 10^{-19}	7.710	0.964749	0.093416 x 10^{-1}
^{250}Fm	^{230}U	^{20}O	38.660	47.996410	5.97889 x 10^{-19}	7.560	1.510719	0.091110 x 10^{-1}
^{251}Fm	^{241}U	^{20}O	38.350	48.746927	5.29829 x 10^{-19}	7.430	1.996054	0.089168 x 10^{-1}
^{252}Fm	^{232}U	^{20}O	38.410	48.522229	5.53139 x 10^{-19}	7.160	3.068491	0.085332 x 10^{-1}
^{253}Fm	^{233}U	^{20}O	33.640	49.123456	6.19591 x 10^{-19}	2.210	2.845056	0.085929 x 10^{-1}
^{254}Fm	^{234}U	^{20}O	38.960	46.952906	7.24323 x 10^{-19}	7.310	2.422269	0.087271 x 10^{-1}
^{255}Fm	^{235}U	^{20}O	39.080	46.379207	1.22614 x 10^{-18}	7.240	2.689300	0.086217 x 10^{-1}
^{256}Fm	^{236}U	^{20}O	39.240	45.904276	1.33177 x 10^{-18}	7.030	3.549478	0.083304 x 10^{-1}
^{257}Fm	^{237}U	^{20}O	39.400	45.432274	1.45730 x 10^{-18}	6.870	4.229231	0.081130 x 10^{-1}

Table 2

Parent	Daughter	Cluster	Q (MeV)	$log_{10}T_{1/2}$	Formation Probability	Q_α (MeV)	$log_{10}T_{1/2}$ (α)	Foramtion Probability(α)
^{177}Tl	^{176}Hg	^{1}H	1.160	-1.670000	-	7.070	-1.945973	7.49481 x 10^{-4}

Table 3

VII. CONCLUSION

Table 1 shows formalism works great even for the cluster in which we had considered it as a point particle. It has been stated that in super heavy nuclei the dominant decay process is cluster decay over alpha decay. It is natural to assume that for heavy nuclei to stabilize heavy clusters should be emitted. But from our result Table 2 shows that alpha decay always dominates over cluster decay even for the super heavy nuclei. From Table 2 we infer that the half-life of the cluster is much larger than the half-life of alpha with a much larger factor. From time scale if there is an alpha particle count we have to wait billions of years to get the single count of the cluster. Also we can see that the formation probability of a cluster is much less or negligible as compared to the formation probability of an alpha particle. Therefore we can say that alpha decay always dominates over cluster decay. We have also calculated nucleon decay which works fine for our case. The calculated half-life nearly matches with the experimental value which is $log_{10}T_{1/2} = -1.176$. For alpha decay of odd nuclei we have considered the available Angular momentum. Table 3 shows efforts made in calculation of half-life of odd-odd super heavy nuclei which further opens the application of this model in decay from odd parent nucleus with odd cluster. This theoretical Model has given us results over the region where Experimental techniques are yet not able to reach.

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