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FAULT TOLERANT METRIC DIMENSION OF HELM GRAPH

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Abstract
In a graph G, an ordered set $W \subseteq V(G)$ is a resolving set of G if every vertex in the set G can be uniquely determined by its vector of distances to the vertex in W. The cardinality of resolving set with the least number of vertices is the metric dimension ($dim(G)$). If for the resolving set W , $W|\{w\}$ is also a resolving set where *w* belongs to W then W is fault tolerant, and its metric dimension is fault tolerant metric dimension. Herein, we find the fault tolerant metric dimension of helm graph.

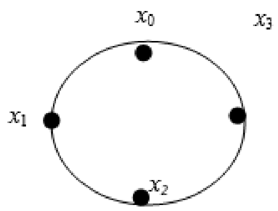
Keywords: Metric dimension, Helm graph, Fault tolerant metric dimension.

Introduction

Slater [17] in the year 1975 and Harary and Melter [7] in the year 1976 originally came up with the term "metric dimension". The distance between two vertices x,y in a graph G is the length of the shortest path in G. Consider $Z = \{z_1, z_2, . . ., z_m\}$ to be an ordered subset of Y and let $y \in Y$. Then we can associate with y an ordered m-tuple that will give the distance from z to all the vertices in Z, denoted by $d(y,Z) = (d(y,z_1), d(y,z_2), \dots, d(y,z_k))$. The set Z is a resolving set of G if for all two vertices $x,y \in Y$, we have $d(x,Z) \neq d(y,Z)$. The cardinality of resolving set with the least number of vertices is the metric dimension ($dim(G)$). The application of fault tolerant metric dimension can be seen in a wide range of systems and networks, including communication networks, power grids, transportation systems, and more. In each case, the goal is to design a system that is resilient to faults or failures, ensuring that critical functions can still be performed even when one or more components of the system fail. One example of the application of fault tolerant metric dimension is in the design and deployment of wireless sensor networks. This idea is being used in fields like mastermind games [5], pharmaceutical chemistry [3], network discovery and verification [2], combinatorial optimization, [16], robot navigation [12].

Hernando et al. [9] in the year 2008 introduced fault tolerance in resolvability. A resolving set Z is fault tolerant if $Z|\{z\}$ is also a resolving set for all $z \in Z$. The minimum cardinality of this resolving set is the fault tolerant metric dimension denoted by $\beta'(G)$. Javaid et al. also c.ontributed to the study of fault-tolerance in resolvability [4,11]. It has also been found that the upper bound is $\beta'(G) \leq dim(G)\ (1 + 2.5\beta(G)-1)$ and lower bound as $\beta'(G) \geq dim(G) + 1$ [9]

Saha et al. [15] found the fault tolerant metric dimension of circulant graphs. Fault tolerant metric dimension of some families of ladder networks [10], fault tolerant metric structure for some crystal structures [13], fault-tolerant metric dimension of $P(n,2)\odot K_1$ graph[1], fault-tolerant metric dimension of interconnection networks [8], fault-tolerant metric dimension of cube of paths[14], fault-tolerant resolvability in some classes of subdivision graphs[6], are some of the researches done till date



Consider a cycle C_4 as shown in Fig.1. Suppose $Z = \{x_0, x_1\}$ then $r(x_0|Z) = (0,1), r(x_1|Z) = (1,0), r(x_2|Z) = (2,1), r(x_3|Z) = (1,2)$. Hence $Z = \{x_0, x_1\}$ is also a resolving set. It is not possible to get a resolving set with just one element in Z. Hence $Z = \{x_0, x_1\}$ is the minimum resolving set, thus $dim(C_4) = 2$.

Suppose $W = \{x_0, x_1, x_2\}$.

$Z_1 = Z|\{x_2\}$ then $r(x_0|Z_1) = (0,1), r(x_1|Z_1) = (1,0), r(x_2|Z_1) = (2,1), r(x_3|Z_1) = (1,2)$

$Z_2 = Z|\{x_1\}$ then $r(x_0|Z_2) = (0,2), r(x_1|Z_2) = (1,1), r(x_2|Z_2) = (2,0), r(x_3|Z_2) = (1,1)$

$Z_3 = Z|\{x_0\}$ then $r(x_0|Z_3) = (1,2), r(x_1|Z_3) = (0,1), r(x_2|Z_3) = (2,0), r(x_3|Z_3) = (2,1)$

In $Z_2, r(x_1|Z_2) = (1,1), r(x_3|Z_2) = (1,1)$. Hence, Z cannot be a fault tolerant resolving set. Thus $\beta'(C_4) \neq 3$.

The fault tolerant metric dimension $\beta'(C_4) = 4$, where $Z = \{x_0, x_1, x_2, x_3\}$.

If $Z_1 = Z|\{x_3\}$ then $r(x_0|Z_1) = (0,1,2), r(x_1|Z_1) = (1,0,1), r(x_2|Z_1) = (2,1,0)$, $r(x_3|Z_1) = (1,2,1)$.

If $Z_2 = Z|\{x_2\}$ then $r(x_0|Z_2) = (0,1,1), r(x_1|Z_2) = (1,0,2), r(x_2|Z_2) = (2,1,1)$, $r(x_3|Z_2) = (1,2,0)$.

If $Z_3 = Z|\{x_1\}$ then $r(x_0|Z_3) = (0,2,1), r(x_1|Z_3) = (1,1,2), r(x_2|Z_3) = (2,0,1)$, $r(x_3|Z_3) = (1,1,0)$.

If $Z_4 = Z|\{x_0\}$ then $r(x_0|Z_4) = (1,2,1), r(x_1|Z_4) = (0,1,2), (x_2|Z_4) = (1,0,1)$, $r(x_3|Z_4) = (2,1,0)$.

MAIN RESULTS

Helm graph:

The helm graph H_n (Fig.3.) is obtained from a wheel graph W_n (Fig.2.)by adjoining a pendant edge at each terminal vertices.

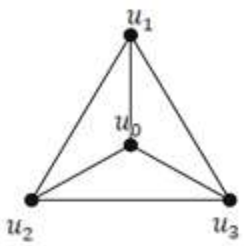


Fig.2. Wheel graph W_3.

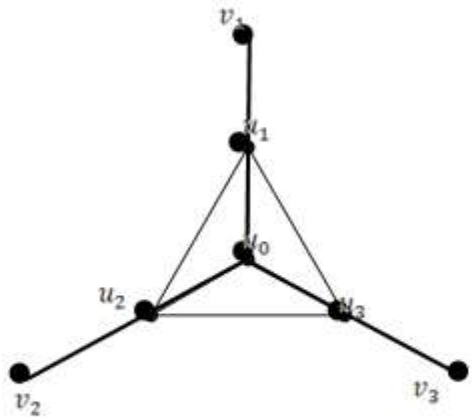


Fig.3. Helm graph H_3.

LEMMA 1:

$$dim(H_n) = \begin{cases} 3, & n = 3,6,7,8 \\ \frac{n-1}{2}, & n \geq 9 \text{ and } n \text{ is odd,} \\ \frac{(n-2)}{2}, & n \geq 10 \text{ and } n \text{ is even,} \end{cases}$$

Since the lower bound of fault tolerant metric dimension is $\beta'(G) \geq dim(G) + 1$, the following lemma is true.

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LEMMA 2 : For $n \geq 6$ $\beta'(H_n) \geq \begin{cases} \frac{n+1}{2}, n \text{ is odd} \\ \frac{n}{2}, n \text{ is even} \end{cases}$

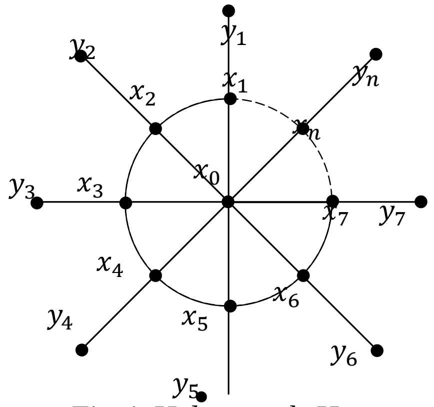

Fig.4. Helm graph H_n.

For a helm graph H_n , let the vertex at the center be x_0. The vertices adjacent with the center x_0 be $x_1,x_2,x_3,\ldots,x_n$ in anticlockwise direction.The pendent vertices adjacent to the terminal vertices $x_1,x_2,x_3,\ldots,x_n$ be $y_1,y_2,y_3,\ldots,y_n$ in anticlockwise direction. The generalized helm graph H_n is shown in Fig.4.

LEMMA 3: For $3 \leq n \leq 5$, $\beta'(H_n) = 4$
Proof: Consider the helm graph H_3. Let $Z = \{x_0, y_1, y_2, y_3\}$. The representation of each vertex of H_3 is shown in the Fig.5.

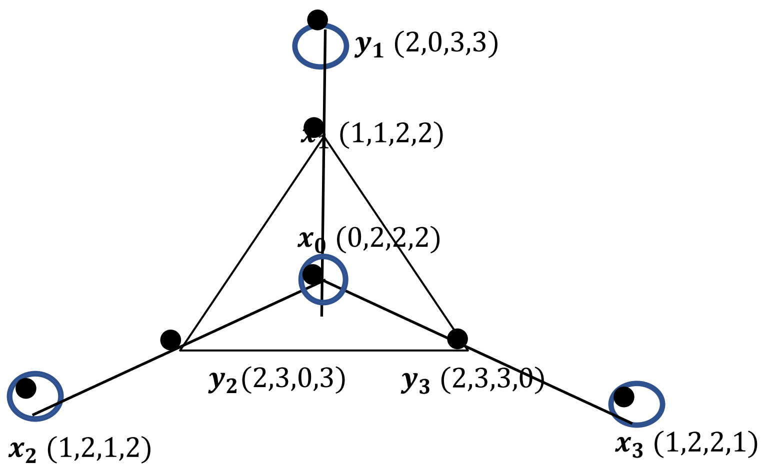

Fig.5. Helm graph H_3 with representation.

For $x \in Z, Z|\{x\}$ is a resolving set for H_3. Thus $\beta'(H_3) = 4$.

Consider the helm graph H_4. Let $Z = \{y_1, y_2, y_3, y_4\}$. The representation of each vertex of H_4 is given below

$$r(x_0|Z) = (2,2,2,2), r(x_1|Z) = (1,2,3,2), r(x_2|Z) = (2,1,2,3),$$
$$r(x_3|Z) = (3,2,1,2), r(x_4|Z) = (2,3,2,1), r(y_1|Z) = (0,3,4,3),$$
$$r(y_2|Z) = (3,0,3,4), r(y_3|Z) = (4,3,0,3), r(y_4|Z) = (3,4,3,0).$$

For $x \in Z, Z|\{x\}$ is a resolving set for H_4. Thus $\beta'(H_4) = 4$.

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Consider the helm graph H_5. Let $Z = \{y_1, y_2, y_3, y_4\}$. The representation of each vertex of H_5 is given below

$$r(x_0|Z) = (2,2,2,2), r(x_1|Z) = (1,2,3,3), r(x_2|Z) = (2,1,2,3),$$
$$r(x_3|Z) = (3,2,1,2), r(x_4|Z) = (3,3,2,1), r(x_5|Z) = (2,3,3,2),$$
$$r(y_1|Z) = (0,3,4,4), r(y_2|Z) = (3,0,3,4), r(y_3|Z) = (4,3,0,3),$$
$$r(y_4|Z) = (4,4,3,0), r(y_5|Z) = (3,4,4,3).$$

For $x \in Z, Z|\{x\}$ is a resolving set for H_5. Thus $\beta'(H_5) = 4$.

THEOREM 1: For $n \geq 7$, $\beta'(H_n) = \frac{(n+1)}{2}$, for odd cases.

Proof: Let H_n , $n \geq 7$ be a helm graph for odd cases. Let $W = \{y_1, y_3, y_5, \dots, y_{n-4}, y_{n-2}, y_n\}$ be the set of bases to resove the graph H_n.Then the representation of the vertices of H_n is given as follows

$r(x_0|Z) = (2,2,2, \dots ,2,2,2), r(x_1|Z) = (1,3,3, \dots ,3,3,2), r(x_n|Z) = (2,3,3, \dots ,3,3,1),$
$r(y_1|Z) = (0,4,4, \dots ,4,4,3), r(y_n|Z) = (3,4,4, \dots ,4,4,0).$

For $1 \leq i \leq \dfrac{(n-1)}{2}$ i^{th} place is 2, $i + 1^{th}$ place is 2, and the remaining places 3 of x_{2i}

$r(x_2|Z) = (2,2,3, \dots ,3,3,3), r(x_4|Z) = (3,2,2, \dots ,3,3,3), r(x_6|Z) = (3,3,2, \dots ,3,3,3), \dots ,$
$r(x_{2i}|Z) = (3,3,3, \dots ,3,2,2).$

For $2 \leq i \leq \frac{(n-1)}{2}$ i^{th} place is 1, and the remaining places 3 in the representation of x_{2i-1}

$r(x_3|Z) = (3,1,3,3, \dots ,3,3), r(x_5|Z) = (3,3,1,3, \dots ,3,3), r(x_7|Z) = (3,3,3,1, \dots ,3,3), \dots ,$
$r(x_{2i-1}|Z) = (3,3,3,3 \dots ,3,1).$

For $1 \leq i \leq \dfrac{n-1}{2}$ i^{th} place is 3, $i + 1^{th}$ place is 3 and the remaining places 4 in the representation of y_{2i}

$r(y_2|Z) = (3,3,4, \dots ,4,4,4), r(y_4|Z) = (4,3,3, \dots ,4,4,4), r(y_6|Z) = (4,4,3, \dots ,4,4,4), \dots,$
$r(y_{2i}|Z) = (4,4,4, \dots ,4,3,3).$

For $2 \leq i \leq \dfrac{n-1}{2}$ i^{th} place is 0, and the remaining places 4 in the representation of y_{2i-1}

$r(y_3|Z) = (4,0,4, \dots ,4,4,4), r(y_5|Z) = (4,0,4, \dots ,4,4,4), r(y_7|Z) = (4,4,0, \dots ,4,4,4), \dots ,$
$r(y_{2i-1}|Z) = (4,4,4, \dots ,4,4,0).$

For $x \in Z, Z|\{x\}$ is a resolving set for H_n. Thus for $n \geq 7$, $\beta'(H_n) = \frac{(n+1)}{2}$.

THEOREM 2: For $n \geq 6$, $\beta'(H_n) = \frac{n}{2}$, for even cases.

Proof:

Let H_n be a Helm Graph $\forall\, n \geq 6$ in even cases. Let $Z = \{y_1, y_3, y_5, \dots , y_{n-3}, y_{n-1}\}$ be the set of bases to resove the graph H_n.Then the representation of the vertices is given as follows

$r(x_0|Z) = (2,2,2, \dots ,2,2,2).$

For $1 \leq i \leq \dfrac{n}{2}$ i^{th} place is 2, $i + 1^{th}$ place is 2 and the remaining places 3 in the representation of u_{2i}

$r(x_2|Z) = (2,2,3, \dots ,3,3,3), r(x_4|Z) = (3,2,2, \dots ,3,3,3), r(x_6|Z) = (3,3,2, \dots ,3,3,3), \dots ,$
$r(x_{2i}|Z) = (3,3,3, \dots ,3,2,2), r(x_{2i}|Z) = (2,3,3, \dots ,3,3,2).$

For $1 \leq i \leq \dfrac{n}{2}$ i^{th} place is 1, and the remaining places 3 in the representation of x_{2i-1}

$r(x_1|Z) = (1,3,3, \dots ,3,3,3), r(x_3|Z) = (3,1,3, \dots ,3,3,3), r(x_5|Z) = (3,3,1, \dots ,3,3,3), \dots$
$r(x_{2i-1}|Z) = (3,3,3 \dots ,3,3,1).$

For $1 \leq i \leq \dfrac{n}{2}$ i^{th} place is 3, $i + 1^{th}$ place is 3 and the remaining places 4 in the

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representation of y_{2i}
$r(y_2|Z) = (3,3,4, \ldots ,4,4,4), r(y_4|Z) = (4,3,3, \ldots ,4,4,4), r(y_6|Z) = (4,4,3, \ldots ,4,4,4), \ldots ,$
$r(y_{2i}|Z) = (4,4,4, \ldots ,4,3,3), r(y_{2i}|Z) = (3,4,4, \ldots ,4,4,3).$
For $1 \le i \le \frac{n}{2} i^{th}$ place is 0, and the remaining places 4 in the representation of y_{2i-1}
$r(y_1|Z) = (0,4,4, \ldots ,4,4,4), r(y_3|Z) = (4,0,4, \ldots ,4,4,4), r(y_5|Z) = (4,4,0, \ldots ,4,4,4), \ldots ,$
$r(y_{2i-1}|Z) = (4,4,4, \ldots ,4,4,0).$
For $x \in Z, Z|\{x\}$ is a resolving set for H_n. Thus for $n \ge 6$, $\beta'(H_n) = \frac{n}{2}$.

Conclusion:

Graph theory is an exceptionally broad field for programmers, designer and engineers. Graphs can be used to solve even very complex prolems. Fault tolerant metric dimension is a complex and challenging problem that requires innovative solutions. Fault tolerant metric dimension is an important tool for ensuring the reliability and resilience of critical systems and networks. By designing systems and networks with redundant paths or components, we can ensure that these systems and networks can continue to function effectively even in the presence of faults or failures. This approach has the potential to be applied in various domains, such as wireless sensor networks, Internet of Things, and social networks. In this paper, the fault tolerant metric dimension of a helm graph H_n , $n \ge 6$ has been found as $\frac{(n+1)}{2}$ if n is odd and $\frac{n}{2}$ if n is even.

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