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ON INTERVAL-VALUED NEUTROSOPHIC FUZZY POSITIVE IMPLICATIVE IDEALS OF BCK-ALGEBRA

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ABSTRACT
The primary objective of this research is to extend the notion of interval-valued neutrosophic fuzzy sets (IVNFS's) to positive implicative ideals (PII's) in BCK-algebra ($\mathcal{BCK}-\mathcal{A}$), thereby introducing interval-valued neutrosophic fuzzy positive implicative ideals (IVNFPII's) of BCK-algebra and we establish relationships between interval-valued neutrosophic fuzzy ideals, interval-valued neutrosophic fuzzy implicative ideals, interval-valued neutrosophic fuzzy commutative ideals & interval-valued neutrosophic fuzzy positive implicative ideals of BCK-algebra and also investigating their fundamental properties.
Keywords: BCK-algebra($\mathcal{BCK}-\mathcal{A}$), Interval-Valued Neutrosophic Fuzzy Sets, Fuzzy Positive Implicative Ideals, Interval-Valued Fuzzy Positive Implicative Ideals, Neutrosophic Interval-Valued Fuzzy Positive Implicative Ideals.
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I. Introduction
BCI/BCK-algebra is a structure of Universal algebras. BCI/BCK-algebraic system developed by Iseki [7, 8] in 1966, 1978, further extends Imai & Iseki [6]. In 1976, Iseki & Tanaka [9] developed the ideal theory of BCK-algebra. The notion of fuzzy set introduced by Zadeh [14] in 1965, which contained the degree of truth membership value and applied to many Mathematical fields. Xi [13] applied the concept of fuzzy sets to BCK-algebra. Meng et. al., [10] introduced the concept of implicative and positive implicative ideals of BCK-algebras, and several researchers investigated further properties of fuzzy BCK-algebra. Atanassov's [5] introduced new idea of IFS's which contained the degree of membership and the degree of non-membership. Atanassov and Gargov [4] developed the concept of IVNFS's as a combination of IFS's and IVFS's. Neutrosophic logic and sets introduced by Smarandache [12]. In 2017 Satyanarayana et.al., [11] introduced the concept of interval-valued intuitionistic fuzzy positive implicative ideals (IVIFPII's) in BCK-algebras and investigated some related properties. The authors [1] recently, studied interval-valued neutrosophic fuzzy hyper BCK-ideals & implicative hyper BCK-ideals of Hyper BCK-algebras and also we [2, 3] are contributed research on interval-valued neutrosophic fuzzy implicative & commutative ideals of BCK-algebras and also in this work generalized to n-fold logic within the context of BCK-algebras. We inspired in this manner to develop interval-valued neutrosophic fuzzy positive implicative ideal of BCK-algebra.
In this research, we introduce the notion of interval-valued neutrosophic fuzzy positive implicative ideals (IVNFPII's) in BCK-algebra ($\mathcal{BCK}-\mathcal{A}$), to establish relationships among interval-valued neutrosophic fuzzy (implicative, commutative and positive implicative) ideals of BCK-algebra and examine their fundamental properties.
For the sake of brevity, we employ the following abbreviations in this paper:
- ✓ $\mathcal{BCK}-\mathcal{A}$ (or) $\mathfrak{A}$: BCK-algebra
- ✓ IVIFS : interval-valued intuitionistic fuzzy set
- ✓ IVIFI : interval-valued intuitionistic fuzzy ideals
- ✓ IVIFPII : interval-valued intuitionistic fuzzy positive implicative ideals

- ✓ NFSA : neutrosophic fuzzy sub-algebra
- ✓ NFI : neutrosophic fuzzy ideals
- ✓ IVNFS : interval-valued neutrosophic fuzzy set
- ✓ IVNFI : interval-valued neutrosophic fuzzy ideals
- ✓ IVNFII : interval-valued neutrosophic fuzzy implicative ideals
- ✓ IVNFPII : interval-valued neutrosophic fuzzy positive implicative ideals
- ✓ IVNFCI : interval-valued neutrosophic fuzzy commutative ideals

II. Preliminaries

Subsequent section, we employed a brief overview of the basic notions required for this study. For the remainder of this paper, $\mathfrak{A}$ will refer to a $\mathcal{BCK}-\mathcal{A}$, unless a different meaning is explicitly stated.

Definition 2.1.[8] Let $\mathfrak{A}$ be a $(\neq \emptyset)$ set with " $\between$ " be a binary operation and " 0 " be a constant. Then $(\mathfrak{A}, \between, 0)$ is said to be a $\mathcal{BCK}-\mathcal{A}$, if it fulfill the listed properties

$(\boldsymbol{\mathcal{BCK}1})$ $\big((\mathfrak{f} \between \mathfrak{g}) \between (\mathfrak{f} \between \hbar)\big) \between (\hbar \between \mathfrak{g}) = 0$,

$(\boldsymbol{\mathcal{BCK}2})$ $\big(\mathfrak{f} \between (\mathfrak{f} \between \mathfrak{g})\big) \between \mathfrak{g} = 0$,

$(\boldsymbol{\mathcal{BCK}3})$ $\mathfrak{f} \between \mathfrak{f} = 0$,

$(\boldsymbol{\mathcal{BCK}4})$ $0 \between \mathfrak{f} = 0$,

$(\boldsymbol{\mathcal{BCK}5})$ $\mathfrak{f} \between \mathfrak{g} = 0$ and $\mathfrak{g} \between \mathfrak{f} = 0 \Rightarrow \mathfrak{f} = \mathfrak{g}$, for any $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

On the set $\mathfrak{A}$, we introduce " $\leq$ " is a binary relation, defined as listed:

$\mathfrak{f} \leq \mathfrak{g} \Leftrightarrow \mathfrak{f} \between \mathfrak{g} = 0$.

In a $\mathcal{BCK}-\mathcal{A}$ $(\mathfrak{A}, \between, 0)$, we have the listed holds true:

$(\boldsymbol{\mathcal{P}1})$ $\mathfrak{f} \between 0 = \mathfrak{f}$,

$(\boldsymbol{\mathcal{P}2})$ $\mathfrak{f} \between \mathfrak{g} \leq \mathfrak{f}$,

$(\boldsymbol{\mathcal{P}3})$ $(\mathfrak{f} \between \mathfrak{g}) \between \hbar = (\mathfrak{f} \between \hbar) \between \mathfrak{g}$,

$(\boldsymbol{\mathcal{P}4})$ $(\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar) \leq \mathfrak{f} \between \mathfrak{g}$,

$(\boldsymbol{\mathcal{P}5})$ $\mathfrak{f} \between \big(\mathfrak{f} \between (\mathfrak{f} \between \mathfrak{g})\big) = \mathfrak{f} \between \mathfrak{g}$,

$(\boldsymbol{\mathcal{P}6})$ $\mathfrak{f} \leq \mathfrak{g} \Rightarrow \mathfrak{f} \between \hbar \leq \mathfrak{g} \between \hbar$ and $\hbar \between \mathfrak{g} \leq \hbar \between \mathfrak{f}$,

$(\boldsymbol{\mathcal{P}7})$ $\mathfrak{f} \between \mathfrak{g} \leq \hbar \Rightarrow \mathfrak{f} \between \hbar \leq \mathfrak{g}$, $\forall$ $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

A $\mathcal{BCK}-\mathcal{A}$ is called a positive implicative, if $(\mathfrak{f} \between \mathfrak{g}) \between \hbar = (\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)$, for all $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

A $(\neq \emptyset)$ sub-set $\mathfrak{I}$ of $\mathfrak{A}$ is said to be a sub-algebra of $\mathfrak{A}$, if $\mathfrak{f} \between \mathfrak{g} \in \mathfrak{I}$, when-ever $\mathfrak{f}, \mathfrak{g} \in \mathfrak{I}$,
an ideal of $\mathfrak{A}$, if $(\mathfrak{I}_1)$ $0 \in \mathfrak{I}$ and $(\mathfrak{I}_2)$ $\mathfrak{f} \between \mathfrak{g}$ and $\mathfrak{g} \in \mathfrak{I} \Rightarrow \mathfrak{f} \in \mathfrak{I}$, $\forall$ $\mathfrak{f}, \mathfrak{g} \in \mathfrak{A}$,
an implicative ideal, if $(\mathfrak{I}_1)$ and $(\mathfrak{I}_3)$ $\big(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})\big) \between \hbar \in \mathfrak{I}$ and $\hbar \in \mathfrak{I}$ implies that $\mathfrak{f} \in \mathfrak{I}$,
for all $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$

Positive implicative ideal, if $(\mathfrak{I}_1)$ and $(\mathfrak{I}_4)$ $(\mathfrak{f} \between \mathfrak{g}) \between \hbar \in \mathfrak{I}$ and $\mathfrak{g} \between \hbar \in \mathfrak{I} \Rightarrow \mathfrak{f} \between \hbar \in \mathfrak{I}$,
for all $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

An IVIFS " $\widetilde{\mathfrak{M}}$ " over $\mathfrak{A}$ is an object having the form $\widetilde{\mathfrak{M}} = \Big\{\Big(\mathfrak{f}, \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f})\Big) : \mathfrak{f} \in \mathfrak{A}\Big\}$, where $\tilde{\xi}_{\mathfrak{M}}: \mathfrak{A} \to \mathbb{A}[0,1]$, and $\tilde{\varpi}_{\mathfrak{M}}: \mathfrak{A} \to \mathbb{A}[0,1]$, the intervals $\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f})$ and $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f})$ denotes the intervals of truth membership degree, and falsity membership degree of the element $\mathfrak{f}$ to the set $\widetilde{\mathfrak{M}}$, where $\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) = [\xi^-_{\mathfrak{M}}(\mathfrak{f}), \xi^+_{\mathfrak{M}}(\mathfrak{f})]$, and $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) = [\varpi^-_{\mathfrak{M}}(\mathfrak{f}), \varpi^+_{\mathfrak{M}}(\mathfrak{f})]$, for all $\mathfrak{f} \in \mathfrak{A}$ with the condition $[0, 0] \leq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) + \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq [1, 1]$, for all $\mathfrak{f} \in \mathfrak{A}$, here $\mathbb{A}[0,1]$ is the set of all closed sub-intervals of $[0,1]$.

An IVNFS " $\widetilde{\mathfrak{M}}$ " over $\mathfrak{A}$ is an object having the form $\widetilde{\mathfrak{M}} = \Big\{\Big(\mathfrak{f}, \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f})\Big) : \mathfrak{f} \in \mathfrak{A}\Big\}$, where $\tilde{\xi}_{\mathfrak{M}}: \mathfrak{A} \to \mathbb{A}[0,1]$, $\tilde{\zeta}_{\mathfrak{M}}: \mathfrak{A} \to \mathbb{A}[0,1]$ and $\tilde{\varpi}_{\mathfrak{M}}: \mathfrak{A} \to \mathbb{A}[0,1]$, the intervals $\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f})$, $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f})$ represents same as on above and $\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f})$ represents indeterminate membership degree of the element $\mathfrak{f}$ to the set $\widetilde{\mathfrak{M}}$, here $\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) = [\xi^-_{\mathfrak{M}}(\mathfrak{f}), \xi^+_{\mathfrak{M}}(\mathfrak{f})]$, $\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) = [\zeta^-_{\mathfrak{M}}(\mathfrak{f}), \zeta^+_{\mathfrak{M}}(\mathfrak{f})]$ and $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) = [\varpi^-_{\mathfrak{M}}(\mathfrak{f}), \varpi^+_{\mathfrak{M}}(\mathfrak{f})]$, for all

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$\mathfrak{f} \in \mathfrak{A}$ with the condition $[0,0] \leq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) + \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) + \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq [1,1]$, for all $\mathfrak{f} \in \mathfrak{A}$. Our convenience, we use the symbol $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$, here $\mathbb{A}[0,1]$ represents same as above one.

Proposition 2.2.[10] In $\mathfrak{A}$, the listed are holds, for all $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$,

i. $\big((\mathfrak{f} \between \hbar) \between \hbar\big) \between (\mathfrak{g} \between \hbar) \leq (\mathfrak{f} \between \mathfrak{g}) \between \hbar$

ii. $(\mathfrak{f} \between \hbar) \between \big(\mathfrak{f} \between (\mathfrak{f} \between \hbar)\big) = (\mathfrak{f} \between \hbar) \between \hbar$

iii. $\Big(\mathfrak{f} \between \big(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})\big)\Big) \between \bigg(\mathfrak{g} \between \Big(\mathfrak{f} \between \big(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})\big)\Big)\bigg) \leq \mathfrak{f} \between \mathfrak{g}$.

Definition 2.3.[8] An Interval-Valued Intuitionistic (IVIFS) $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ in $\mathfrak{A}$ is an Interval-Valued Intuitionistic Fuzzy Positive Implicative ideal (IVIFPII) of $\mathfrak{A}$, if it fulfills

$(\boldsymbol{IVIFPI_1})$ $\tilde{\xi}_{\mathfrak{M}}(0) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f})$, and $\tilde{\varpi}_{\mathfrak{M}}(0) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f})$,

$(\boldsymbol{IVIFPI_2})$ $\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f} \between \hbar) \geq \min\{\tilde{\xi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g} \between \hbar)\}$,

$(\boldsymbol{IVIFPI_3})$ $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between \hbar) \leq \max\{\tilde{\varpi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g} \between \hbar)\}$, for all $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

Definition 2.4.[3] An Interval-Valued Neutrosophic Fuzzy Set (IVNFS) $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ in $\mathfrak{A}$ is an Interval-Valued Neutrosophic Fuzzy Ideal (IVNFI) of $\mathfrak{A}$, if it satisfies

$(\boldsymbol{IVNFI_1})$ $\tilde{\xi}_{\mathfrak{M}}(0) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f})$, $\tilde{\zeta}_{\mathfrak{M}}(0) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f})$ and $\tilde{\varpi}_{\mathfrak{M}}(0) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f})$,

$(\boldsymbol{IVNFI_2})$ $\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\{\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})\}$,

$(\boldsymbol{IVNFI_3})$ $\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\{\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})\}$,

$(\boldsymbol{IVNFI_4})$ $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\{\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})\}$, for all $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

Definition 2.5.[2] An interval-valued neutrosophic fuzzy set (IVNFS) $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ in $\mathfrak{A}$ is an Interval-Valued Neutrosophic Fuzzy Implicative Ideal (IVNFII) of $\mathfrak{A}$, if it fulfills

$(\boldsymbol{IVNFII_1})$ $\tilde{\xi}_{\mathfrak{M}}(0) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f})$, $\tilde{\zeta}_{\mathfrak{M}}(0) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f})$ and $\tilde{\varpi}_{\mathfrak{M}}(0) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f})$,

$(\boldsymbol{IVNFII_2})$ $\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\Big\{\tilde{\xi}_{\mathfrak{M}}\Big(\big(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})\big) \between \hbar\Big), \tilde{\xi}_{\mathfrak{M}}(\hbar)\Big\}$,

$(\boldsymbol{IVNFII_3})$ $\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\Big\{\tilde{\zeta}_{\mathfrak{M}}\Big(\big(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})\big) \between \hbar\Big), \tilde{\zeta}_{\mathfrak{M}}(\hbar)\Big\}$,

$(\boldsymbol{IVNFII_4})$ $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\Big\{\tilde{\varpi}_{\mathfrak{M}}\Big(\big(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})\big) \between \hbar\Big), \tilde{\varpi}_{\mathfrak{M}}(\hbar)\Big\}$, for all $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

Definition 2.6[2]. An interval-valued neutrosophic fuzzy set (IVNFS) $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ in $\mathfrak{A}$ is an Interval-Valued Neutrosophic Fuzzy Commutative Ideal (IVNFCI) of $\mathfrak{A}$, if it fulfills

$(\boldsymbol{IVNFCI_1})$ $\tilde{\xi}_{\mathfrak{M}}(0) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f})$, $\tilde{\zeta}_{\mathfrak{M}}(0) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f})$ and $\tilde{\varpi}_{\mathfrak{M}}(0) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f})$,

$(\boldsymbol{IVNFCI_2})$ $\tilde{\xi}_{\mathfrak{M}}\Big(\mathfrak{f} \between \big(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})\big)\Big) \geq \min\{\tilde{\xi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar), \tilde{\xi}_{\mathfrak{M}}(\hbar)\}$,

$(\boldsymbol{IVNFCI_3})$ $\tilde{\zeta}_{\mathfrak{M}}\Big(\mathfrak{f} \between \big(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})\big)\Big) \geq \min\{\tilde{\zeta}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar), \tilde{\zeta}_{\mathfrak{M}}(\hbar)\}$,

$(\boldsymbol{IVNFCI_4})$ $\tilde{\varpi}_{\mathfrak{M}}\Big(\mathfrak{f} \between \big(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})\big)\Big) \leq \max\{\tilde{\varpi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar), \tilde{\varpi}_{\mathfrak{M}}(\hbar)\}$, for all $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

Theorem 2.7.[2] Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ be an interval-valued neutrosophic fuzzy ideal (IVNFI) of $\mathfrak{A}$, if $\mathfrak{f} \leq \mathfrak{g}$ in $\mathfrak{A}$, then $\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g})$, $\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g})$ and $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g})$, **i.e.**, $\tilde{\xi}_{\mathfrak{M}}$, $\tilde{\zeta}_{\mathfrak{M}}$ and $\tilde{\varpi}_{\mathfrak{M}}$ are reversing order and preserving order.

Theorem 2.8.[2] An interval-valued neutrosophic fuzzy subalgebra (IVNFSA) $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ is a neutrosophic fuzzy ideal (NFI) of $\mathfrak{A}$ $\Leftrightarrow$ for $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$, $\mathfrak{f} \between \mathfrak{g} \leq \hbar \Rightarrow \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\{\tilde{\xi}_{\mathfrak{M}}(\mathfrak{g}), \tilde{\xi}_{\mathfrak{M}}(\hbar)\}$, $\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \min\{\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g}), \tilde{\zeta}_{\mathfrak{M}}(\hbar)\}$ and $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \max\{\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g}), \tilde{\varpi}_{\mathfrak{M}}(\hbar)\}$.

Theorem 2.9.[2] Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ be an interval-valued neutrosophic fuzzy ideal (IVNFI) of $\mathfrak{A}$. Then listed are interchangeable.

i. $\widetilde{\mathfrak{M}}$ is an interval-valued neutrosophic fuzzy implicative ideal (IVNFII) of $\mathfrak{A}$.

ii. $\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) \geq \tilde{\xi}_{\mathfrak{M}}\big(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})\big)$, $\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \geq \tilde{\zeta}_{\mathfrak{M}}\big(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})\big)$ and $\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) \leq \tilde{\varpi}_{\mathfrak{M}}\big(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})\big)$, $\forall$ $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

iii. $\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) = \tilde{\xi}_{\mathfrak{M}}\big(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})\big)$, $\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) = \tilde{\zeta}_{\mathfrak{M}}\big(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})\big)$ and

$\tilde{\varpi}_{\mathfrak{M}}(\mathbb{f}) = \tilde{\varpi}_{\mathfrak{M}}(\mathbb{f} \between (\mathbb{g} \between \mathbb{f})), \forall\ \mathbb{f}, \mathbb{g}, \hbar \in \mathfrak{A}$.

Theorem 2.10.[2] Let $\mathfrak{A}$ be an Implicative BCK-algebra; then every interval-valued neutrosophic fuzzy ideals (IVNFI) of $\mathfrak{A}$ is an interval-valued neutrosophic fuzzy implicative ideals (IVNFII) of $\mathfrak{A}$.

III. Interval-Valued Neutrosophic Fuzzy Positive Implicative Ideals in BCK-algebra

In this section we developed the concept of interval-valued neutrosophic fuzzy positive implicative ideals (IVNFPII) in $\mathfrak{A}$ and also proved their related definition, counter examples and theorems.

Definition 3.1. An interval-valued neutrosophic fuzzy set (IVNFS) $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ in $\mathfrak{A}$ is an interval-valued neutrosophic fuzzy positive implicative ideals (IVNFPII) of $\mathfrak{A}$, if it fulfills

$(\boldsymbol{IVNFPI_1})$ $\tilde{\xi}_{\mathfrak{M}}(0) \geq \tilde{\xi}_{\mathfrak{M}}(\mathbb{f})$, $\tilde{\zeta}_{\mathfrak{M}}(0) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathbb{f})$ and $\tilde{\varpi}_{\mathfrak{M}}(0) \leq \tilde{\varpi}_{\mathfrak{M}}(\mathbb{f})$,

$(\boldsymbol{IVNFPI_2})$ $\tilde{\xi}_{\mathfrak{M}}(\mathbb{f} \between \hbar) \geq \min\{\tilde{\xi}_{\mathfrak{M}}((\mathbb{f} \between \mathbb{g}) \between \hbar), \tilde{\xi}_{\mathfrak{M}}(\mathbb{g} \between \hbar)\}$,

$(\boldsymbol{IVNFPI_3})$ $\tilde{\zeta}_{\mathfrak{M}}(\mathbb{f} \between \hbar) \geq \min\{\tilde{\zeta}_{\mathfrak{M}}((\mathbb{f} \between \mathbb{g}) \between \hbar), \tilde{\zeta}_{\mathfrak{M}}(\mathbb{g} \between \hbar)\}$,

$(\boldsymbol{IVNFPI_4})$ $\tilde{\varpi}_{\mathfrak{M}}(\mathbb{f} \between \hbar) \leq \max\{\tilde{\varpi}_{\mathfrak{M}}((\mathbb{f} \between \mathbb{g}) \between \hbar), \tilde{\varpi}_{\mathfrak{M}}(\mathbb{g} \between \hbar)\}$, for all $\mathbb{f}, \mathbb{g}, \hbar \in \mathfrak{A}$.

Example 3.2. Consider $\mathfrak{A} = \{0, q_1, q_2, q_3\}$ in which "$\between$" is defined in the listed Cayley table

$\between$	0	q_1	q_2	q_3
0	0	0	0	0
q_1	q_1	0	q_1	0
q_2	q_2	q_2	0	0
q_3	q_3	q_2	q_1	0

Then $(\mathfrak{A}, \between, 0)$ is a $\mathcal{BCK} - \mathcal{A}$. Define an IVNFS $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ in $\mathfrak{A}$ by

$\tilde{\xi}_{\mathfrak{M}}(0) = \tilde{l}_0$, $\tilde{\xi}_{\mathfrak{M}}(q_1) = \tilde{l}_1$, $\tilde{\xi}_{\mathfrak{M}}(q_2) = \tilde{\xi}_{\mathfrak{M}}(q_3) = \tilde{l}_2$, $\tilde{\zeta}_{\mathfrak{M}}(0) = \tilde{r}_0$, $\tilde{\zeta}_{\mathfrak{M}}(q_1) = \tilde{r}_1$,
$\tilde{\zeta}_{\mathfrak{M}}(q_2) = \tilde{\zeta}_{\mathfrak{M}}(q_3) = \tilde{r}_2$ and $\tilde{\varpi}_{\mathfrak{M}}(0) = \tilde{n}_0$, $\tilde{\varpi}_{\mathfrak{M}}(q_1) = \tilde{n}_1$, $\tilde{\varpi}_{\mathfrak{M}}(q_2) = \tilde{\varpi}_{\mathfrak{M}}(q_3) = \tilde{n}_2$,
where $\tilde{l}_0 > \tilde{l}_1 > \tilde{l}_2$, $\tilde{r}_0 > \tilde{r}_1 > \tilde{r}_2$ and $\tilde{n}_0 < \tilde{n}_1 < \tilde{n}_2$ and $[0, 0] \leq \tilde{l}_i + \tilde{r}_i + \tilde{n}_i \leq [1, 1]$ for $i = 0,1,2$. Then $\widetilde{\mathfrak{M}}$ is an IVNFPII of $\mathfrak{A}$.

Theorem 3.3. An interval-valued neutrosophic fuzzy positive implicative ideals (IVNFPII) $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ of $\mathfrak{A}$ must be an interval-valued neutrosophic fuzzy ideals (IVNFI).

In general, converse of Theorem 3.3 may not be hold good, see the below counter example.

Example 3.4. Consider $\mathfrak{A} = \{0, q_1, q_2, q_3, q_4\}$ with the Cayley table below

$\between$	0	q_1	q_2	q_3	q_4
0	0	0	0	0	0
q_1	q_1	0	q_1	0	0
q_2	q_2	q_2	0	0	0
q_3	q_3	q_3	q_3	0	0
q_4	q_4	q_4	q_4	q_3	0

Define an IVNFS $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ in $\mathfrak{A}$ by

$\tilde{\xi}_{\mathfrak{M}}(0) = \tilde{l}_0$, $\tilde{\xi}_{\mathfrak{M}}(q_1) = \tilde{l}_1$, $\tilde{\xi}_{\mathfrak{M}}(q_2) = \tilde{\xi}_{\mathfrak{M}}(q_3) = \tilde{\xi}_{\mathfrak{M}}(q_4) = \tilde{l}_2$,
$\tilde{\zeta}_{\mathfrak{M}}(0) = \tilde{r}_0$, $\tilde{\zeta}_{\mathfrak{M}}(q_1) = \tilde{r}_1$, $\tilde{\zeta}_{\mathfrak{M}}(q_2) = \tilde{\zeta}_{\mathfrak{M}}(q_3) = \tilde{\zeta}_{\mathfrak{M}}(q_4) = \tilde{r}_2$ and
$\tilde{\varpi}_{\mathfrak{M}}(0) = \tilde{n}_0$, $\tilde{\varpi}_{\mathfrak{M}}(q_1) = \tilde{n}_1$, $\tilde{\varpi}_{\mathfrak{M}}(q_2) = \tilde{\varpi}_{\mathfrak{M}}(q_3) = \tilde{\varpi}_{\mathfrak{M}}(q_4) = \tilde{n}_2$,
where $\tilde{l}_0 > \tilde{l}_1 > \tilde{l}_2$, $\tilde{r}_0 > \tilde{r}_1 > \tilde{r}_2$ and $\tilde{n}_0 < \tilde{n}_1 < \tilde{n}_2$ and $[0, 0] \leq \tilde{l}_i + \tilde{r}_i + \tilde{n}_i \leq [1, 1]$, for $i = 0,1,2$.
Simple calculations gives that $\widetilde{\mathfrak{M}}$ is an IVNFI of $\mathfrak{A}$, but it is not an IVNFPII of $\mathfrak{A}$, because

$\tilde{\xi}_{\mathfrak{M}}(q_4 \between q_3) = \tilde{l}_2 < \tilde{l}_0 = \min\{\tilde{\xi}_{\mathfrak{M}}((q_4 \between q_3) \between q_3), \tilde{\xi}_{\mathfrak{M}}(q_3 \between q_3)\}$,
$\tilde{\zeta}_{\mathfrak{M}}(q_4 \between q_3) = \tilde{r}_2 < \tilde{r}_0 = \min\{\tilde{\zeta}_{\mathfrak{M}}((q_4 \between q_3) \between q_3), \tilde{\zeta}_{\mathfrak{M}}(q_3 \between q_3)\}$ and
$\tilde{\varpi}_{\mathfrak{M}}(q_4 \between q_3) = \tilde{n}_2 > \tilde{n}_0 = \max\{\tilde{\varpi}_{\mathfrak{M}}((q_4 \between q_3) \between q_3), \tilde{\varpi}_{\mathfrak{M}}(q_3 \between q_3)\}$.

Theorem 3.5. Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ be an IVNFPII of $\mathfrak{A}$, then $\tilde{\xi}_{\mathfrak{M}}$, $\tilde{\zeta}_{\mathfrak{M}}$ and $\tilde{\varpi}_{\mathfrak{M}}$ are order reversing and preserving.

Proof: This theorem proof follows from Definition 3.1 & Theorem 2.7.

Theorem 3.6. If $\mathfrak{A}$ is positive implicative BCK-algebra (PI-$\mathcal{BCK}$- $\mathcal{A}$), then an IVNFI must be an IVNFPII.

Proof: Follows from Theorem 2.10.

Proposition 3.7. Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}})$ be an IVNFI of $\mathfrak{A}$. Then
$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}) \geq \min\{\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f} \between \hbar), \tilde{\xi}_{\mathfrak{M}}(\hbar \between \mathfrak{g})\}$,
$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}) \geq \min\{\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f} \between \hbar), \tilde{\zeta}_{\mathfrak{M}}(\hbar \between \mathfrak{g})\}$ and
$\widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}) \leq \max\{\widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between \hbar), \widetilde{\varpi}_{\mathfrak{M}}(\hbar \between \mathfrak{g})\}$, $\forall\ \mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

Proposition 3.8. Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}})$ be an IVNFI of $\mathfrak{A}$, then $\widetilde{\mathfrak{M}}$ is an IVNFPII of $\mathfrak{A} \Leftrightarrow$
$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}) \geq \tilde{\xi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \mathfrak{g})$, $\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}) \geq \tilde{\zeta}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \mathfrak{g})$ and $\widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}) \leq \widetilde{\varpi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \mathfrak{g})$,
$\forall\ \mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

Theorem 3.9. Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}})$ be an IVNFI of $\mathfrak{A}$. Then
$\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}})$ is an IVNFPII of $\mathfrak{A} \Leftrightarrow \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}) = \tilde{\xi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \mathfrak{g})$,
$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}) = \tilde{\zeta}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \mathfrak{g})$ and $\widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}) = \widetilde{\varpi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \mathfrak{g})$, $\forall\ \mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

Theorem 3.10. Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}})$ be an IVNFI of $\mathfrak{A}$. Then
$\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}})$ is an IVNFCI of $\mathfrak{A} \Leftrightarrow \tilde{\xi}_{\mathfrak{M}}\left(\mathfrak{f} \between \left(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})\right)\right) = \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g})$,
$\tilde{\zeta}_{\mathfrak{M}}\left(\mathfrak{f} \between \left(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})\right)\right) = \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g})$ and $\widetilde{\varpi}_{\mathfrak{M}}\left(\mathfrak{f} \between \left(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})\right)\right) = \widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g})$, $\forall\ \mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

Theorem 3.11. An IVNFI $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}})$ of $\mathfrak{A}$ is an IVNFII $\Leftrightarrow \widetilde{\mathfrak{M}}$ is both an IVNFCI and an IVNFPII.

Proof: Suppose that $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}})$ is an IVNFII of $\mathfrak{A}$. By (2.2(i) and 2.8), we have

$$\begin{aligned}\min\{\tilde{\xi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g} \between \hbar)\} &\leq \tilde{\xi}_{\mathfrak{M}}((\mathfrak{f} \between \hbar) \between \hbar) \\ &= \tilde{\xi}_{\mathfrak{M}}\left((\mathfrak{f} \between \hbar) \between \left(\mathfrak{f} \between (\mathfrak{f} \between \hbar)\right)\right) && \text{(by 2.2(ii))} \\ &= \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f} \between \hbar) && \text{(by 2.9(iii))}\end{aligned}$$

$$\begin{aligned}\min\{\tilde{\zeta}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g} \between \hbar)\} &\leq \tilde{\zeta}_{\mathfrak{M}}((\mathfrak{f} \between \hbar) \between \hbar) \\ &= \tilde{\zeta}_{\mathfrak{M}}\left((\mathfrak{f} \between \hbar) \between \left(\mathfrak{f} \between (\mathfrak{f} \between \hbar)\right)\right) && \text{(by 2.2(ii))} \\ &= \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f} \between \hbar) && \text{(by 2.9(iii))}\end{aligned}$$

and

$$\begin{aligned}\max\{\widetilde{\varpi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar), \widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{g} \between \hbar)\} &\geq \widetilde{\varpi}_{\mathfrak{M}}((\mathfrak{f} \between \hbar) \between \hbar) \\ &= \widetilde{\varpi}_{\mathfrak{M}}\left((\mathfrak{f} \between \hbar) \between \left(\mathfrak{f} \between (\mathfrak{f} \between \hbar)\right)\right) && \text{(by 2.2(ii))} \\ &= \widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between \hbar) && \text{(by 2.9(iii))}\end{aligned}$$

Hence, $\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f} \between \hbar) \geq \min\{\tilde{\xi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g} \between \hbar)\}$,
$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f} \between \hbar) \geq \min\{\tilde{\zeta}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g} \between \hbar)\}$ and
$\widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between \hbar) \leq \max\{\widetilde{\varpi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar), \widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{g} \between \hbar)\}$, $\forall\ \mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.
Therefore, $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}})$ is an IVNFPII of $\mathfrak{A}$.
And by Theorem's 2.7, 2.9(iii) and 2.2(iii), we get

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}) \leq \tilde{\xi}_{\mathfrak{M}}\left(\mathfrak{f} \between \left(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})\right)\right) \between \left(\mathfrak{g} \between \left(\mathfrak{f} \between \left(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})\right)\right)\right) = \tilde{\xi}_{\mathfrak{M}}\left(\mathfrak{f} \between \left(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})\right)\right),$$

$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}) \leq \tilde{\zeta}_{\mathfrak{M}}\left(\mathfrak{f} \between \left(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})\right)\right) \between \left(\mathfrak{g} \between \left(\mathfrak{f} \between \left(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})\right)\right)\right) = \tilde{\zeta}_{\mathfrak{M}}\left(\mathfrak{f} \between \left(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})\right)\right) \text{ and}$$

$$\widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}) \geq \widetilde{\varpi}_{\mathfrak{M}}\left(\mathfrak{f} \between \left(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})\right)\right) \between \left(\mathfrak{g} \between \left(\mathfrak{f} \between \left(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})\right)\right)\right) = \widetilde{\varpi}_{\mathfrak{M}}\left(\mathfrak{f} \between \left(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})\right)\right).$$

It follows that $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}})$ is an IVNFCI of $\mathfrak{A}$.
Conversely, assume that $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}})$ is both IVNFPII and IVNFCI of $\mathfrak{A}$.

Since, $(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})) \between (\mathfrak{g} \between \mathfrak{f}) \leq \mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f}), \forall\, \mathfrak{f}, \mathfrak{g} \in \mathfrak{A}$. It follows that

$$\tilde{\xi}_{\mathfrak{M}}\left((\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})) \between (\mathfrak{g} \between \mathfrak{f})\right) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})),$$
$$\tilde{\zeta}_{\mathfrak{M}}\left((\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})) \between (\mathfrak{g} \between \mathfrak{f})\right) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})) \text{ and}$$
$$\widetilde{\varpi}_{\mathfrak{M}}\left((\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})) \between (\mathfrak{g} \between \mathfrak{f})\right) \leq \widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})).$$

Using, Theorem 3.8[2], we get

$$\tilde{\xi}_{\mathfrak{M}}\left((\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})) \between (\mathfrak{g} \between \mathfrak{f})\right) = \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})),$$
$$\tilde{\zeta}_{\mathfrak{M}}\left((\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})) \between (\mathfrak{g} \between \mathfrak{f})\right) = \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})) \text{ and}$$
$$\widetilde{\varpi}_{\mathfrak{M}}\left((\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})) \between (\mathfrak{g} \between \mathfrak{f})\right) = \widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})).$$

Hence,

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})) \leq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})) \leq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})) \text{ and}$$
$$\widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})) \geq \widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f})).....(i)$$

On the other hand, since $\mathfrak{f} \between \mathfrak{g} \leq \mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})$, we have

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}) \geq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}) \geq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})) \text{ and}$$
$$\widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}) \leq \widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})).$$

Since, $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}})$ is an IVNFCI of $\mathfrak{A}$, then by Theorem 3.10, we have

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}) = \tilde{\xi}_{\mathfrak{M}}\left(\mathfrak{f} \between (\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f}))\right),$$
$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}) = \tilde{\zeta}_{\mathfrak{M}}\left(\mathfrak{f} \between (\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f}))\right) \text{ and}$$
$$\widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}) = \widetilde{\varpi}_{\mathfrak{M}}\left(\mathfrak{f} \between (\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f}))\right).$$

Therefore,

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})) = \tilde{\xi}_{\mathfrak{M}}\left(\mathfrak{f} \between (\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f}))\right),$$
$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})) = \tilde{\zeta}_{\mathfrak{M}}\left(\mathfrak{f} \between (\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f}))\right) \text{ and}$$
$$\widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})) = \widetilde{\varpi}_{\mathfrak{M}}\left(\mathfrak{f} \between (\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f}))\right)..... (ii)$$

Combining, (i), (ii) and $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}})$ be a NFI of $\mathfrak{A}$, we get

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})) \leq \min\left\{\tilde{\xi}_{\mathfrak{M}}\left(\mathfrak{f} \between (\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f}))\right), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f}))\right\} \leq \tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}),$$
$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})) \leq \min\left\{\tilde{\zeta}_{\mathfrak{M}}\left(\mathfrak{f} \between (\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f}))\right), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f}))\right\} \leq \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) \text{ and}$$
$$\widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \mathfrak{f})) \geq \max\left\{\widetilde{\varpi}_{\mathfrak{M}}\left(\mathfrak{f} \between (\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f}))\right), \widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{g} \between (\mathfrak{g} \between \mathfrak{f}))\right\} \geq \widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}).$$

By Theorem 2.9, $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}})$ is an IVNFII of $\mathfrak{A}$.

Theorem 3.12. Let $\mathfrak{I} \subseteq \mathfrak{A}$ and $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}})$ be an IVNFS in $\mathfrak{A}$ defined by

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) = \begin{cases} \tilde{\gamma}_0, & \text{if } \mathfrak{f} \in \mathfrak{I} \\ \tilde{\gamma}_1, & \text{otherwise} \end{cases}, \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) = \begin{cases} \tilde{\delta}_0, & \text{if } \mathfrak{f} \in \mathfrak{I} \\ \tilde{\delta}_1, & \text{otherwise} \end{cases} \text{ and } \widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) = \begin{cases} \tilde{\sigma}_0, & \text{if } \mathfrak{f} \in \mathfrak{I} \\ \tilde{\sigma}_1, & \text{otherwise} \end{cases},$$

$\forall\, \mathfrak{f} \in \mathfrak{A}$, where $0 \leq \tilde{\gamma}_1 < \tilde{\gamma}_0$, $0 \leq \tilde{\delta}_1 < \tilde{\delta}_0$, $0 \leq \tilde{\sigma}_0 < \tilde{\sigma}_1$ and $\tilde{\gamma}_i + \tilde{\delta}_i + \tilde{\sigma}_i \leq \tilde{1}$, where $\tilde{\gamma}_i = [\gamma_i^-, \gamma_i^+]$, $\tilde{\delta}_i = [\delta_i^-, \delta_i^+]$, $\tilde{\sigma}_i = [\sigma_i^-, \sigma_i^+]$ for $i = 0,1$. Then the listed conditions are interchangeable:

i. $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}})$ is an IVNFII of $\mathfrak{A}$.

ii. $\mathfrak{I}$ is an II of $\mathfrak{A}$.

Corollary 3.13. Let $\mathfrak{I} \subseteq \mathfrak{A}$ and $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}})$ be a NFS in $\mathfrak{A}$ defined by

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f}) = \begin{cases} 1, & \text{if } \mathfrak{f} \in \mathfrak{I} \\ 0, & \text{otherwise} \end{cases}, \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f}) = \begin{cases} 1, & \text{if } \mathfrak{f} \in \mathfrak{I} \\ 0, & \text{otherwise} \end{cases} \text{ and } \widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{f}) = \begin{cases} 0, & \text{if } \mathfrak{f} \in \mathfrak{I} \\ 1, & \text{otherwise} \end{cases},$$

$\forall\ \mathbb{f} \in \mathfrak{A}$. Then the listed conditions are interchangeable:

i. $\widetilde{\mathfrak{M}} = \left(\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}}\right)$ is an IVNFII of $\mathfrak{A}$.

ii. $\mathfrak{I}$ is an II of $\mathfrak{A}$.

Theorem 3.14. $\mathfrak{A}$ is implicative $\Leftrightarrow$ every IVNFI of $\mathfrak{A}$ is an IVNFII.

Proof: Suppose that $\mathfrak{A}$ is an implicative-BCK-A(I-$\mathcal{BCK}$- $\mathcal{A}$).

Assume that $\widetilde{\mathfrak{M}} = \left(\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}}\right)$ is an IVNFI of $\mathfrak{A}$, then for any $s, t, v \in [0,1]$, $\mathcal{U}\left(\tilde{\xi}_{\mathfrak{M}}; \tilde{s}\right)$, $\mathcal{U}\left(\tilde{\zeta}_{\mathfrak{M}}; \tilde{t}\right)$ and $\mathcal{L}\left(\tilde{\varpi}_{\mathfrak{M}}; \tilde{v}\right)$ are either empty (or) ideals of $\mathfrak{A}$. And so for any $\tilde{s}, \tilde{t}, \tilde{v} \in \mathbb{A}[0,1]$, $\mathcal{U}\left(\tilde{\xi}_{\mathfrak{M}}; \tilde{s}\right)$, $\mathcal{U}\left(\tilde{\zeta}_{\mathfrak{M}}; \tilde{t}\right)$ and $\mathcal{L}\left(\tilde{\varpi}_{\mathfrak{M}}; \tilde{v}\right)$ are either empty (or) II's of $\mathfrak{A}$. By Theorem 3.12[2], we have $\widetilde{\mathfrak{M}} = \left(\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}}\right)$ is an IVNFII of $\mathfrak{A}$.

Conversely, suppose that in an implicative BCK-algebra, every IVNFI of $\mathfrak{A}$ is an IVNFII.

Let $\mathfrak{I}$ be an ideal of $\mathfrak{A}$. NFS $\widetilde{\mathfrak{M}} = \left(\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}}\right)$ defined by

$$\tilde{\xi}_{\mathfrak{M}}(\mathbb{f}) = \begin{cases} \tilde{\gamma}_0, & \text{if } \mathbb{f} \in \mathfrak{I} \\ \tilde{\gamma}_1, & \text{otherwise} \end{cases}, \quad \tilde{\zeta}_{\mathfrak{M}}(\mathbb{f}) = \begin{cases} \tilde{\delta}_0, & \text{if } \mathbb{f} \in \mathfrak{I} \\ \tilde{\delta}_1, & \text{otherwise} \end{cases} \text{ and } \tilde{\varpi}_{\mathfrak{M}}(\mathbb{f}) = \begin{cases} \tilde{\sigma}_0, & \text{if } \mathbb{f} \in \mathfrak{I} \\ \tilde{\sigma}_1, & \text{otherwise} \end{cases}$$

$\forall\ \mathbb{f} \in \mathfrak{A}$, where $0 \le \tilde{\gamma}_1 < \tilde{\gamma}_0$, $0 \le \tilde{\delta}_1 < \tilde{\delta}_0$, $0 \le \tilde{\sigma}_0 < \tilde{\sigma}_1$ and $\tilde{\gamma}_i + \tilde{\delta}_i + \tilde{\sigma}_i \le \tilde{1}$,
where $\tilde{\gamma}_i = [\gamma_i^-, \gamma_i^+]$, $\tilde{\delta}_i = [\delta_i^-, \delta_i^+]$, $\tilde{\sigma}_i = [\sigma_i^-, \sigma_i^+]$ for $i = 0,1$.

It is easy to see that $\widetilde{\mathfrak{M}} = \left(\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}}\right)$ is an IVNFI of $\mathfrak{A}$. By converse hypothesis,
$\widetilde{\mathfrak{M}} = \left(\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}}\right)$ is an IVNFII of $\mathfrak{A}$. We have $\mathfrak{I}$ is an implicative ideal of $\mathfrak{A}$.

This shows that every ideal of $\mathfrak{A}$ is an implicative ideal of $\mathfrak{A}$.

Corollary 3.15. For a $\mathcal{BCK} - \mathcal{A}$ the listed conditions are interchangeable :

i. $\mathfrak{A}$ is implicative.

ii. Every ideal of $\mathfrak{A}$ is implicative.

iii. Every IVNFI of $\mathfrak{A}$ is an IVNFII.

iv. Every IVNFI of $\mathfrak{A}$ is both an IVNFCI and IVNFPII.

Similarly, we can prove the cases of IVNFCI and IVNFPII of $\mathfrak{A}$. We omit the proof.

Theorem 3.16. If $\widetilde{\mathfrak{M}} = \left(\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}}\right)$ is an IVNFI of $\mathfrak{A}$ with the listed conditions hold

i. $\tilde{\xi}_{\mathfrak{M}}(\mathbb{f} \between \mathbb{g}) \ge \min\left\{\tilde{\xi}_{\mathfrak{M}}\left(((\mathbb{f} \between \mathbb{g}) \between \mathbb{g}) \between \hbar\right), \tilde{\xi}_{\mathfrak{M}}(\hbar)\right\}$,

ii. $\tilde{\zeta}_{\mathfrak{M}}(\mathbb{f} \between \mathbb{g}) \ge \min\left\{\tilde{\zeta}_{\mathfrak{M}}\left(((\mathbb{f} \between \mathbb{g}) \between \mathbb{g}) \between \hbar\right), \tilde{\zeta}_{\mathfrak{M}}(\hbar)\right\}$,

iii. $\tilde{\varpi}_{\mathfrak{M}}(\mathbb{f} \between \mathbb{g}) \le \max\left\{\tilde{\varpi}_{\mathfrak{M}}\left(((\mathbb{f} \between \mathbb{g}) \between \mathbb{g}) \between \hbar\right), \tilde{\varpi}_{\mathfrak{M}}(\hbar)\right\}$, $\forall\ \mathbb{f}, \mathbb{g}, \hbar \in \mathfrak{A}$.

Then $\widetilde{\mathfrak{M}}$ is an IVNFPII of $\mathfrak{A}$.

Proof: Assume that $\widetilde{\mathfrak{M}} = \left(\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}}\right)$ is an IVNFI of $\mathfrak{A}$, with the following conditions.

$$\tilde{\xi}_{\mathfrak{M}}(\mathbb{f} \between \mathbb{g}) \ge \min\left\{\tilde{\xi}_{\mathfrak{M}}\left(((\mathbb{f} \between \mathbb{g}) \between \mathbb{g}) \between \hbar\right), \tilde{\xi}_{\mathfrak{M}}(\hbar)\right\},$$
$$\tilde{\zeta}_{\mathfrak{M}}(\mathbb{f} \between \mathbb{g}) \ge \min\left\{\tilde{\zeta}_{\mathfrak{M}}\left(((\mathbb{f} \between \mathbb{g}) \between \mathbb{g}) \between \hbar\right), \tilde{\zeta}_{\mathfrak{M}}(\hbar)\right\} \text{ and}$$
$$\tilde{\varpi}_{\mathfrak{M}}(\mathbb{f} \between \mathbb{g}) \le \max\left\{\tilde{\varpi}_{\mathfrak{M}}\left(((\mathbb{f} \between \mathbb{g}) \between \mathbb{g}) \between \hbar\right), \tilde{\varpi}_{\mathfrak{M}}(\hbar)\right\}, \forall\ \mathbb{f}, \mathbb{g}, \hbar \in \mathfrak{A}.$$

Using $(\mathcal{P}3)$ and $(\mathcal{P}4)$, we have

$$\left(((\mathbb{f} \between \hbar) \between \hbar) \between (\mathbb{g} \between \hbar)\right) \le (\mathbb{f} \between \hbar) \between \mathbb{g} = (\mathbb{f} \between \mathbb{g}) \between \hbar, \forall\ \mathbb{f}, \mathbb{g}, \hbar \in \mathfrak{A}.$$

Hence, by Theorem 3.6, we get,

$$\tilde{\xi}_{\mathfrak{M}}\left(((\mathbb{f} \between \hbar) \between \hbar) \between (\mathbb{g} \between \hbar)\right) \ge \tilde{\xi}_{\mathfrak{M}}\left((\mathbb{f} \between \mathbb{g}) \between \hbar\right),$$
$$\tilde{\zeta}_{\mathfrak{M}}\left(((\mathbb{f} \between \hbar) \between \hbar) \between (\mathbb{g} \between \hbar)\right) \ge \tilde{\zeta}_{\mathfrak{M}}\left((\mathbb{f} \between \mathbb{g}) \between \hbar\right) \text{ and}$$
$$\tilde{\varpi}_{\mathfrak{M}}\left(((\mathbb{f} \between \hbar) \between \hbar) \between (\mathbb{g} \between \hbar)\right) \le \tilde{\varpi}_{\mathfrak{M}}\left((\mathbb{f} \between \mathbb{g}) \between \hbar\right).$$

It follows from hypothesis, we get

$$\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f} \between \hbar) \geq \min\left\{\tilde{\xi}_{\mathfrak{M}}\left(((\mathfrak{f} \between \hbar) \between \hbar) \between (\mathfrak{g} \between \hbar)\right), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g} \between \hbar)\right\}$$
$$\geq \min\{\tilde{\xi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g} \between \hbar)\},$$
$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f} \between \hbar) \geq \min\left\{\tilde{\zeta}_{\mathfrak{M}}\left(((\mathfrak{f} \between \hbar) \between \hbar) \between (\mathfrak{g} \between \hbar)\right), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g} \between \hbar)\right\}$$
$$\geq \min\{\tilde{\zeta}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g} \between \hbar)\},$$

and

$$\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between \hbar) \leq \max\left\{\tilde{\varpi}_{\mathfrak{M}}\left(((\mathfrak{f} \between \hbar) \between \hbar) \between (\mathfrak{g} \between \hbar)\right), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g} \between \hbar)\right\}$$
$$\leq \max\{\tilde{\varpi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g} \between \hbar)\}, \forall\, \mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}.$$

Therefore, $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ is an IVNFPII of $\mathfrak{A}$.

Theorem 3.17. Let $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ be an IVNFI of $\mathfrak{A}$, then $\widetilde{\mathfrak{M}}$ is an IVNFPII of $\mathfrak{A} \Leftrightarrow$

i. $\tilde{\xi}_{\mathfrak{M}}((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)) \geq \tilde{\xi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar)$,

ii. $\tilde{\zeta}_{\mathfrak{M}}((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)) \geq \tilde{\zeta}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar)$ and

iii. $\tilde{\varpi}_{\mathfrak{M}}((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)) \leq \tilde{\varpi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar)$, $\forall\, \mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

Proof: Assume that $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ is an IVNFI of $\mathfrak{A}$ and fulfills the below conditions

$$\tilde{\xi}_{\mathfrak{M}}((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)) \geq \tilde{\xi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar),$$
$$\tilde{\zeta}_{\mathfrak{M}}((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)) \geq \tilde{\zeta}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar) \text{ and}$$
$$\tilde{\varpi}_{\mathfrak{M}}((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)) \leq \tilde{\varpi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar), \forall\, \mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}.$$

Hence, $\tilde{\xi}_{\mathfrak{M}}(\mathfrak{f} \between \hbar) \geq \min\{\tilde{\xi}_{\mathfrak{M}}((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g} \between \hbar)\}$

$$\geq \min\{\tilde{\xi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{g} \between \hbar)\},$$
$$\tilde{\zeta}_{\mathfrak{M}}(\mathfrak{f} \between \hbar) \geq \min\{\tilde{\zeta}_{\mathfrak{M}}((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g} \between \hbar)\}$$
$$\geq \min\{\tilde{\zeta}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{g} \between \hbar)\},$$

and

$$\tilde{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between \hbar) \leq \max\{\tilde{\varpi}_{\mathfrak{M}}((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g} \between \hbar)\}$$
$$\leq \max\{\tilde{\varpi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar), \tilde{\varpi}_{\mathfrak{M}}(\mathfrak{g} \between \hbar)\}, \forall\, \mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}.$$

Therefore, $\widetilde{\mathfrak{M}}$ is an IVNFPII of $\mathfrak{A}$.

Conversely, suppose that $\widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ is an IVNFPII of $\mathfrak{A} \Rightarrow \widetilde{\mathfrak{M}} = (\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \tilde{\varpi}_{\mathfrak{M}})$ is an IVNFI of $\mathfrak{A}$. Let $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$ be such that $\mathfrak{p} = \mathfrak{f} \between (\mathfrak{g} \between \hbar)$ and $\mathfrak{q} = \mathfrak{f} \between \mathfrak{g}$,

since, $\left((\mathfrak{f} \between (\mathfrak{g} \between \hbar)) \between (\mathfrak{f} \between \mathfrak{g})\right) \leq \mathfrak{g} \between (\mathfrak{g} \between \hbar)$, we have

$$\tilde{\xi}_{\mathfrak{M}}((\mathfrak{p} \between \mathfrak{q}) \between \hbar) = \tilde{\xi}_{\mathfrak{M}}\left(\left((\mathfrak{f} \between (\mathfrak{g} \between \hbar)) \between (\mathfrak{f} \between \mathfrak{g})\right) \between \hbar\right)$$
$$\geq \tilde{\xi}_{\mathfrak{M}}\left((\mathfrak{g} \between (\mathfrak{g} \between \hbar)) \between \hbar\right)$$
$$= \tilde{\xi}_{\mathfrak{M}}(0) \quad [\text{By } (\mathcal{BCK}1), (\mathcal{BCK}3) \text{ and } (\mathcal{P}3)]$$

and so, $\tilde{\xi}_{\mathfrak{M}}((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)) = \tilde{\xi}_{\mathfrak{M}}\left((\mathfrak{f} \between (\mathfrak{g} \between \hbar)) \between \hbar\right)$

$$= \tilde{\xi}_{\mathfrak{M}}(\mathfrak{p} \between \hbar)$$
$$\geq \min\{\tilde{\xi}_{\mathfrak{M}}((\mathfrak{p} \between \mathfrak{q}) \between \hbar), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{p} \between \hbar)\}$$
$$\geq \min\{\tilde{\xi}_{\mathfrak{M}}(0), \tilde{\xi}_{\mathfrak{M}}(\mathfrak{p} \between \hbar)\}$$
$$= \tilde{\xi}_{\mathfrak{M}}(\mathfrak{p} \between \hbar)$$
$$= \tilde{\xi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar).$$

Hence, $\tilde{\xi}_{\mathfrak{M}}((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)) \geq \tilde{\xi}_{\mathfrak{M}}((\mathfrak{f} \between \mathfrak{g}) \between \hbar)$, $\forall\, \mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

$$\tilde{\zeta}_{\mathfrak{M}}((\mathfrak{p} \between \mathfrak{q}) \between \hbar) = \tilde{\zeta}_{\mathfrak{M}}\left(\left((\mathfrak{f} \between (\mathfrak{g} \between \hbar)) \between (\mathfrak{f} \between \mathfrak{g})\right) \between \hbar\right)$$
$$\geq \tilde{\zeta}_{\mathfrak{M}}\left((\mathfrak{g} \between (\mathfrak{g} \between \hbar)) \between \hbar\right)$$

$$= \tilde{\zeta}_{\mathfrak{M}}(0) \quad [\text{By } (\mathcal{BCK}1), (\mathcal{BCK}3) \text{ and } (\mathcal{P}3)]$$

and so, $\tilde{\zeta}_{\mathfrak{M}}\big((\mathbb{f} \between \hbar) \between (\mathfrak{g} \between \hbar)\big) = \tilde{\zeta}_{\mathfrak{M}}\Big(\big(\mathbb{f} \between (\mathfrak{g} \between \hbar)\big) \between \hbar\Big)$

$$\begin{aligned}
&= \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{p} \between \hbar) \\
&\geq \min\{\tilde{\zeta}_{\mathfrak{M}}\big((\mathfrak{p} \between \mathbb{q}) \between \hbar\big), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{p} \between \hbar)\} \\
&\geq \min\{\tilde{\zeta}_{\mathfrak{M}}(0), \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{p} \between \hbar)\} \\
&= \tilde{\zeta}_{\mathfrak{M}}(\mathfrak{p} \between \hbar) \\
&= \tilde{\zeta}_{\mathfrak{M}}\big((\mathbb{f} \between \mathfrak{g}) \between \hbar\big).
\end{aligned}$$

Hence, $\tilde{\zeta}_{\mathfrak{M}}\big((\mathbb{f} \between \hbar) \between (\mathfrak{g} \between \hbar)\big) \geq \tilde{\zeta}_{\mathfrak{M}}\big((\mathbb{f} \between \mathfrak{g}) \between \hbar\big), \forall\, \mathbb{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

$$\begin{aligned}
\widetilde{\varpi}_{\mathfrak{M}}\big((\mathfrak{p} \between \mathbb{q}) \between \hbar\big) &= \widetilde{\varpi}_{\mathfrak{M}}\Big(\Big(\big(\mathbb{f} \between (\mathfrak{g} \between \hbar)\big) \between (\mathbb{f} \between \mathfrak{g})\Big) \between \hbar\Big) \\
&\leq \widetilde{\varpi}_{\mathfrak{M}}\Big(\big(\mathfrak{g} \between (\mathfrak{g} \between \hbar)\big) \between \hbar\Big) \\
&= \widetilde{\varpi}_{\mathfrak{M}}(0) \quad [\text{By } (\mathcal{BCK}1), (\mathcal{BCK}3) \text{ and } (\mathcal{P}3)]
\end{aligned}$$

and so, $\widetilde{\varpi}_{\mathfrak{M}}\big((\mathbb{f} \between \hbar) \between (\mathfrak{g} \between \hbar)\big) = \widetilde{\varpi}_{\mathfrak{M}}\Big(\big(\mathbb{f} \between (\mathfrak{g} \between \hbar)\big) \between \hbar\Big)$

$$\begin{aligned}
&= \widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{p} \between \hbar) \\
&\leq \max\{\widetilde{\varpi}_{\mathfrak{M}}\big((\mathfrak{p} \between \mathbb{q}) \between \hbar\big), \widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{p} \between \hbar)\} \\
&\leq \max\{\widetilde{\varpi}_{\mathfrak{M}}(0), \widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{p} \between \hbar)\} \\
&= \widetilde{\varpi}_{\mathfrak{M}}(\mathfrak{p} \between \hbar) \\
&= \widetilde{\varpi}_{\mathfrak{M}}\big((\mathbb{f} \between \mathfrak{g}) \between \hbar\big).
\end{aligned}$$

Hence, $\widetilde{\varpi}_{\mathfrak{M}}\big((\mathbb{f} \between \hbar) \between (\mathfrak{g} \between \hbar)\big) \leq \widetilde{\varpi}_{\mathfrak{M}}\big((\mathbb{f} \between \mathfrak{g}) \between \hbar\big), \forall\, \mathbb{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

Therefore, $\tilde{\xi}_{\mathfrak{M}}\big((\mathbb{f} \between \hbar) \between (\mathfrak{g} \between \hbar)\big) \geq \tilde{\xi}_{\mathfrak{M}}\big((\mathbb{f} \between \mathfrak{g}) \between \hbar\big)$,

$\tilde{\zeta}_{\mathfrak{M}}\big((\mathbb{f} \between \hbar) \between (\mathfrak{g} \between \hbar)\big) \geq \tilde{\zeta}_{\mathfrak{M}}\big((\mathbb{f} \between \mathfrak{g}) \between \hbar\big)$ and

$\widetilde{\varpi}_{\mathfrak{M}}\big((\mathbb{f} \between \hbar) \between (\mathfrak{g} \between \hbar)\big) \leq \widetilde{\varpi}_{\mathfrak{M}}\big((\mathbb{f} \between \mathfrak{g}) \between \hbar\big), \forall\, \mathbb{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

Theorem 3.18. Let $\widetilde{\mathfrak{M}} = \big(\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}}\big)$ be an IVNFS of $\mathfrak{A}$. Then $\widetilde{\mathfrak{M}}$ is an IVNFPII of $\mathfrak{A} \Leftrightarrow$ the nonempty upper $\tilde{s}$-level cut $\mathcal{U}\big(\tilde{\xi}_{\mathfrak{M}}; \tilde{s}\,\big)$, the nonempty upper $\tilde{t}$-level cut $\mathcal{U}\big(\tilde{\zeta}_{\mathfrak{M}}; \tilde{t}\,\big)$ and the nonempty lower $\tilde{v}$-level cut $\mathcal{L}(\widetilde{\varpi}_{\mathfrak{M}}; \tilde{v}\,)$ are PII's of $\mathfrak{A}$, for any $\tilde{s}, \tilde{t}, \tilde{v} \in [0, 1]$.

Note: (i) $\widetilde{\mathfrak{M}}(0) = \widetilde{\mathfrak{K}}(0) \Rightarrow \tilde{\xi}_{\mathfrak{M}}(0) = \tilde{\xi}_{\mathfrak{K}}(0), \tilde{\zeta}_{\mathfrak{M}}(0) = \tilde{\zeta}_{\mathfrak{K}}(0), \widetilde{\varpi}_{\mathfrak{M}}(0) = \widetilde{\varpi}_{\mathfrak{K}}(0)$.

(ii) $\widetilde{\mathfrak{M}} \subseteq \widetilde{\mathfrak{K}} \Rightarrow \tilde{\xi}_{\mathfrak{M}}(\mathbb{f}) = \tilde{\xi}_{\mathfrak{K}}(\mathbb{f}), \tilde{\zeta}_{\mathfrak{M}}(\mathbb{f}) = \tilde{\zeta}_{\mathfrak{K}}(\mathbb{f}), \widetilde{\varpi}_{\mathfrak{M}}(\mathbb{f}) = \widetilde{\varpi}_{\mathfrak{K}}(\mathbb{f}), \forall\, \mathbb{f} \in \mathfrak{A}$.

Theorem 3.19. (Extension property for an interval-valued neutrosophic fuzzy positive implicative ideal (IVNFPII))

Let $\widetilde{\mathfrak{M}} = \big(\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}}\big)$ and $\widetilde{\mathfrak{K}} = \big(\tilde{\xi}_{\mathfrak{K}}, \tilde{\zeta}_{\mathfrak{K}}, \widetilde{\varpi}_{\mathfrak{K}}\big)$ be two IVNFI's of $\mathfrak{A}$ such that $\widetilde{\mathfrak{M}}(0) = \widetilde{\mathfrak{K}}(0)$ and $\widetilde{\mathfrak{M}} \subseteq \widetilde{\mathfrak{K}}$. If $\widetilde{\mathfrak{M}}$ is an IVNFPII of $\mathfrak{A}$, then so is $\widetilde{\mathfrak{K}}$.

Proof: Assume that $\widetilde{\mathfrak{M}} = \big(\tilde{\xi}_{\mathfrak{M}}, \tilde{\zeta}_{\mathfrak{M}}, \widetilde{\varpi}_{\mathfrak{M}}\big)$ is an IVNFPII of $\mathfrak{A}$

$\tilde{\xi}_{\mathfrak{K}}\Big(\big((\mathbb{f} \between \hbar) \between (\mathfrak{g} \between \hbar)\big) \between \big((\mathbb{f} \between \mathfrak{g}) \between \hbar\big)\Big)$

$$\begin{aligned}
&= \tilde{\xi}_{\mathfrak{K}}\Big(\Big((\mathbb{f} \between \hbar) \between \big((\mathbb{f} \between \mathfrak{g}) \between \hbar\big)\Big) \between (\mathfrak{g} \between \hbar)\Big) \quad [\text{By } (\mathcal{P}3)] \\
&= \tilde{\xi}_{\mathfrak{K}}\Big(\Big(\Big(\mathbb{f} \between \big((\mathbb{f} \between \mathfrak{g}) \between \hbar\big)\Big) \between \hbar\Big) \between (\mathfrak{g} \between \hbar)\Big) \quad [\text{By } (\mathcal{P}3)] \\
&\geq \tilde{\xi}_{\mathfrak{M}}\Big(\Big(\Big(\mathbb{f} \between \big((\mathbb{f} \between \mathfrak{g}) \between \hbar\big)\Big) \between \hbar\Big) \between (\mathfrak{g} \between \hbar)\Big) \quad [\because \tilde{\xi}_{\mathfrak{M}} \subseteq \tilde{\xi}_{\mathfrak{K}}] \\
&\geq \tilde{\xi}_{\mathfrak{M}}\Big(\Big(\Big(\mathbb{f} \between \big((\mathbb{f} \between \mathfrak{g}) \between \hbar\big)\Big) \between \mathfrak{g}\Big) \between \hbar\Big) \\
&= \tilde{\xi}_{\mathfrak{M}}\Big(\Big((\mathbb{f} \between \hbar) \between \big((\mathbb{f} \between \mathfrak{g}) \between \hbar\big)\Big) \between \hbar\Big) \; [\text{By } (\mathcal{P}3)]
\end{aligned}$$

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$$= \tilde{\xi}_{\mathfrak{M}}\left(\left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right) \between \left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right) \text{ [By } (\mathcal{P}3)]$$

$$= \tilde{\xi}_{\mathfrak{M}}(0) = \tilde{\xi}_{\mathfrak{K}}(0). \qquad \text{[By } (\mathcal{BCK}3) \text{ and Supposition]}$$

It follows from $(IVNFI1)$ and $(IVNFI2)$ that

$$\tilde{\xi}_{\mathfrak{K}}\left((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)\right) \geq \min\left\{\tilde{\xi}_{\mathfrak{K}}\left(\left((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)\right) \between \left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right), \tilde{\xi}_{\mathfrak{K}}\left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right\}$$

$$\geq \min\left\{\tilde{\xi}_{\mathfrak{K}}(0), \tilde{\xi}_{\mathfrak{K}}\left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right\}$$

$$= \tilde{\xi}_{\mathfrak{K}}\left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right), \forall\, \mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}.$$

Hence, $\tilde{\xi}_{\mathfrak{K}}\left((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)\right) \geq \tilde{\xi}_{\mathfrak{K}}\left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)$, for any $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$

$$\tilde{\zeta}_{\mathfrak{K}}\left(\left((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)\right) \between \left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right)$$

$$= \tilde{\zeta}_{\mathfrak{K}}\left(\left((\mathfrak{f} \between \hbar) \between \left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right) \between (\mathfrak{g} \between \hbar)\right) \text{ [By } (\mathcal{P}3)]$$

$$= \tilde{\zeta}_{\mathfrak{K}}\left(\left(\left(\mathfrak{f} \between \left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right) \between \hbar\right) \between (\mathfrak{g} \between \hbar)\right) \text{ [By } (\mathcal{P}3)]$$

$$\geq \tilde{\zeta}_{\mathfrak{M}}\left(\left(\left(\mathfrak{f} \between \left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right) \between \hbar\right) \between (\mathfrak{g} \between \hbar)\right) \quad [\because \tilde{\zeta}_{\mathfrak{M}} \subseteq \tilde{\zeta}_{\mathfrak{K}}]$$

$$\geq \tilde{\zeta}_{\mathfrak{M}}\left(\left(\left(\mathfrak{f} \between \left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right) \between \mathfrak{g}\right) \between \hbar\right)$$

$$= \tilde{\zeta}_{\mathfrak{M}}\left(\left((\mathfrak{f} \between \hbar) \between \left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right) \between \hbar\right) \text{ [By } (\mathcal{P}3)]$$

$$= \tilde{\zeta}_{\mathfrak{M}}\left(\left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right) \between \left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right) \text{ [By } (\mathcal{P}3)]$$

$$= \tilde{\zeta}_{\mathfrak{M}}(0) = \tilde{\zeta}_{\mathfrak{K}}(0). \qquad \text{[By } (\mathcal{BCK}3) \text{ and Supposition]}$$

It follows from $(IVNFI1)$ and $(IVNFI3)$ that

$$\tilde{\zeta}_{\mathfrak{K}}\left((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)\right) \geq \min\left\{\tilde{\zeta}_{\mathfrak{K}}\left(\left((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)\right) \between \left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right), \tilde{\zeta}_{\mathfrak{K}}\left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right\}$$

$$\geq \min\left\{\tilde{\zeta}_{\mathfrak{K}}(0), \tilde{\zeta}_{\mathfrak{K}}\left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right\}$$

$$= \tilde{\zeta}_{\mathfrak{K}}\left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right), \forall\, \mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}.$$

Hence, $\tilde{\zeta}_{\mathfrak{K}}\left((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)\right) \geq \tilde{\zeta}_{\mathfrak{K}}\left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)$, for any $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$, and

$$\tilde{\varpi}_{\mathfrak{K}}\left(\left((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)\right) \between \left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right)$$

$$= \tilde{\varpi}_{\mathfrak{K}}\left(\left((\mathfrak{f} \between \hbar) \between \left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right) \between (\mathfrak{g} \between \hbar)\right) \text{ [By } (\mathcal{P}3)]$$

$$= \tilde{\varpi}_{\mathfrak{K}}\left(\left(\left(\mathfrak{f} \between \left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right) \between \hbar\right) \between (\mathfrak{g} \between \hbar)\right) \text{ [By } (\mathcal{P}3)]$$

$$\leq \tilde{\varpi}_{\mathfrak{M}}\left(\left(\left(\mathfrak{f} \between \left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right) \between \hbar\right) \between (\mathfrak{g} \between \hbar)\right) \quad [\because \tilde{\varpi}_{\mathfrak{M}} \subseteq \tilde{\varpi}_{\mathfrak{K}}]$$

$$\leq \tilde{\varpi}_{\mathfrak{M}}\left(\left(\left(\mathfrak{f} \between \left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right) \between \mathfrak{g}\right) \between \hbar\right)$$

$$= \tilde{\varpi}_{\mathfrak{M}}\left(\left((\mathfrak{f} \between \hbar) \between \left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right) \between \hbar\right) \text{ [By } (\mathcal{P}3)]$$

$$= \tilde{\varpi}_{\mathfrak{M}}\left(\left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right) \between \left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right) \text{ [By } (\mathcal{P}3)]$$

$$= \tilde{\varpi}_{\mathfrak{M}}(0) = \tilde{\varpi}_{\mathfrak{K}}(0). \qquad \text{[By } (\mathcal{BCK}3) \text{ and Supposition]}$$

It follows from $(IVNFI1)$ and $(IVNFI4)$ that

$$\tilde{\varpi}_{\mathfrak{K}}\left((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)\right)$$

$$\leq \max\left\{\tilde{\varpi}_{\mathfrak{K}}\left(\left((\mathfrak{f} \between \hbar) \between (\mathfrak{g} \between \hbar)\right) \between \left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right), \tilde{\varpi}_{\mathfrak{K}}\left((\mathfrak{f} \between \mathfrak{g}) \between \hbar\right)\right\}$$

$$\leq \max\{\tilde{\varpi}_{\mathfrak{K}}(0), \tilde{\varpi}_{\mathfrak{K}}((\mathbb{f} \between \mathfrak{g}) \between \hbar)\}$$
$$= \tilde{\varpi}_{\mathfrak{K}}((\mathbb{f} \between \mathfrak{g}) \between \hbar), \forall\, \mathbb{f}, \mathfrak{g}, \hbar \in \mathfrak{A}.$$

Hence, $\tilde{\varpi}_{\mathfrak{K}}((\mathbb{f} \between \hbar) \between (\mathfrak{g} \between \hbar)) \leq \tilde{\varpi}_{\mathfrak{K}}((\mathbb{f} \between \mathfrak{g}) \between \hbar)$, for any $\mathbb{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.
Therefore, $\tilde{\mathfrak{K}} = (\tilde{\xi}_{\mathfrak{K}}, \tilde{\zeta}_{\mathfrak{K}}, \tilde{\varpi}_{\mathfrak{K}})$ is an IVNFPII of $\mathfrak{A}$.

IV. Conclusion

In this research we successfully discussed the primary notion of interval-valued neutrosophic fuzzy positive implicative ideals (PII) in $\mathfrak{A}$. Further, we discussed the relationships between interval-valued neutrosophic fuzzy ideals (IVNFI), interval-valued neutrosophic fuzzy implicative ideals (IVNFII), interval-valued neutrosophic fuzzy commutative ideals (IVNFCI), and interval-valued neutrosophic fuzzy positive implicative ideals (IVNFPII) of $\mathfrak{A}$, then we given some counter examples and also investigated their fundamental concepts.

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